

AD-A135 146

FINAL REPORT ON SCIENTIFIC ACTIVITIES PURSUANT TO THE  
PROVISIONS OF AFOSR. (U) WISCONSIN UNIV-MADISON DEPT OF  
MATHEMATICS D L RUSSELL JUL 83 AFOSR-TR-83-0881  
AFOSR-79-0018

1/1

UNCLASSIFIED

F/G 14/2

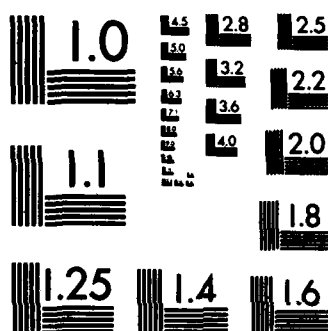
NL

END

FILMED

1-84

DTIC



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

②

| REPORT DOCUMENTATION PAGE                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |                                         | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                                             |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------|-----------------------------------------------------------------------------------------|
| 1. REPORT NUMBER<br><b>AFOSR-TR- 33 - 0881</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | 2. GOVT ACCESSION NO.<br><b>A135146</b> | 3. RECIPIENT'S CATALOG NUMBER                                                           |
| 4. TITLE (and Subtitle)<br><br>"FINAL REPORT ON SCIENTIFIC ACTIVITIES PURSUANT TO<br>THE PROVISIONS OF AFOSR GRANT-79-0018"<br>Nov 1, 1981 through October 31, 1982                                                                                                                                                                                                                                                                                                                                                                                                                              |                                         | 5. TYPE OF REPORT & PERIOD COVERED<br><br>FINAL                                         |
| 7. AUTHOR(s)<br><br>D.L. Russell                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |                                         | 6. PERFORMING ORG. REPORT NUMBER                                                        |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br><br>University of Wisconsin<br>Department of Mathematics<br>Madison WI 53706                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                         | 8. CONTRACT OR GRANT NUMBER(s)<br><br>AFOSR-79-0018                                     |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br><br>AFOSR/NM<br>Bldg. 410<br>Bolling AFB DC 20332                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                         | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS<br><br>PE61102F; 2304/A1 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                         | 12. REPORT DATE<br><br>July 1983                                                        |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                         | 13. NUMBER OF PAGES<br><br>85                                                           |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                         | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED                                |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                         | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                                           |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><br>Approved for public release; distribution unlimited.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                         |                                                                                         |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                         |                                                                                         |
| 18. SUPPLEMENTARY NOTES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                         |                                                                                         |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br><br>The author                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |                                         |                                                                                         |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>We review various aspects of the control theory of hyperbolic systems, includ-<br>ing controllability, stabilization, control canonical form theory, etc. To<br>allow a unified and not excessively technical treatment, we restrict attention<br>to the case of a single space variable; the multi-dimensional case is treated<br>in our more extensive review [36]. The paper concludes with a short dis-<br>cussion of the newly developed procedure of canonical augmentation.<br><br>another work. |                                         |                                                                                         |

A135.146

DTIC FILE COPY

DTIC  
ELECTE  
NOV 30 1983  
S E D  
attention is restricted

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AFOSR-TR- 83 - 0881

FINAL REPORT ON SCIENTIFIC ACTIVITIES  
PURSUANT TO THE PROVISIONS OF  
AFOSR GRANT 79-0018-  
DURING THE PERIOD  
NOVEMBER 1, 1981 TO OCTOBER 31, 1982  
D. L. RUSSELL, PRINCIPAL INVESTIGATOR  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF WISCONSIN  
MADISON



|                    |                                     |
|--------------------|-------------------------------------|
| Accession For      |                                     |
| NTIS GRA&I         | <input checked="" type="checkbox"/> |
| DTIC TAB           | <input type="checkbox"/>            |
| Unannounced        | <input type="checkbox"/>            |
| Justification      |                                     |
| By                 |                                     |
| Distribution/      |                                     |
| Availability Codes |                                     |
| Dist               | Avail and/or<br>Special             |
| A-1                |                                     |

Approved for public release  
distribution unlimited.

83 11 29 255

### 1. General Remarks.

During the period November 1, 1981 to October 31, 1982, the Principal Investigator, in cooperation with several research assistants, carried out a program of mathematical research in the general area of control theory of partial differential equations. The program involved two distinct phases: an effort aimed specifically at the development and improvement of control strategies in connection with the wing flutter problem and a more general program in the area of distributed parameter control problems of hyperbolic type.

This work resulted in two scientific papers which form the greater part of this report. The first of these, "Some Remarks on the Current Status of the Control Theory of Single Space Dimension Hyperbolic Systems" was presented at the NASA JPL Symposium on Control and Stabilization of Large Space Structures, Pasadena, CA, July, 1982. The second, "Admissible Input Elements for Systems in Hilbert Space and a Carleson Measure Criterion", by L. F. Ho and the Principal Investigator, is a paper which largely resulted from Dr. Ho's thesis work, also supported by this grant, in part.

In addition to Dr. Ho, who is now with the University of Iowa, the Principal Investigator was assisted by R. G. Teglus, H. M. Baron, and R. Rebarber..

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)  
NOTICE OF TRANSMITTAL TO DTIC  
This technical report has been reviewed and is  
approved for public release IAW AFR 190-12.  
Distribution is unlimited.  
MATTHEW J. KERPER  
Chief, Technical Information Division

Travel Supported by the Grant.

Grant funds were used to support travel by the Principal Investigator and one Research Assistant, H. M. Baron.

The Principal Investigator travelled to Pasadena, California, and presented a paper at the NASA Jet Propulsion Laboratory Symposium on Control and Stabilization of Large Space Structures. The paper presented concerned the current status of the control of hyperbolic partial differential equations with particular emphasis on observers and canonical structure.

The Principal Investigator also took part in the 30th Anniversary meeting of the Society for Industrial and Applied Mathematics (SIAM) in Palo Alto, California, July, 1982. A paper on the treatment of control problems associated with infinite dimensional linear systems by means of methods from the theory of analytic functions was presented at this meeting by invitation of the organizing committee. Ms. Baron also attended this meeting and presented a paper on control canonical forms for systems governed by various types of partial differential equations.

3.. Technical Appendix.

This appendix consists of two papers whose preparation was supported in part by the grant. These papers are:

"Some Remarks on the Current Status of the Control Theory of Single Space Dimension Hyperbolic Systems"

and

"Admissible Input Elements for Systems in Hilbert Space and a Carleson Measure Criterion",

the latter paper being jointly authored by L. F. Ho and the Principal Investigator.

**SOME REMARKS ON THE CURRENT STATUS OF THE CONTROL THEORY  
OF SINGLE SPACE DIMENSION HYPERBOLIC SYSTEMS**

**D. L. Russell\*\***

**University of Wisconsin Mathematics Department**

**ABSTRACT**

We review various aspects of the control theory of hyperbolic systems, including controllability, stabilization, control canonical form theory, etc. To allow a unified and not excessively technical treatment, we restrict attention to the case of a single space variable; the multi-dimensional case is treated in our more extensive review [36]. The paper concludes with a short discussion of the newly developed procedure of canonical augmentation.

**SOME ASPECTS OF THE CONTROL THEORY OF THE WAVE EQUATION  
AND RELATED SYSTEMS**

The systematic study of control systems governed by partial differential equations, a special, but exceptionally important, subcategory of distributed parameter systems began in the early 1960's with the work of the Soviet scientists A. G. Butkovskii [3], [4], Yu. V. Egorov [11] and others. These works were primarily concerned with the extension of Pontryagin's Maximum Principle [26] to certain classes of processes which could not be satisfactorily modelled by finite dimensional mathematical systems. Controllability questions were raised but were usually subsidiary to questions of optimality. One of the first systematic controllability studies, in connection with the heat equation, was presented by Gal'chuk in [14]. One of the most important of the early American contributions to the subject was the 1963 thesis of Fattorini [13], which also treated parabolic systems and was one of the first works to recognize the strong relationship between distributed parameter control studies

\* Supported in part by the Air Force Office of Scientific Research under Grant AFOSR 79-0018.

\*\* Dept. of Mathematics, University of Wisconsin, Madison, WI 53706. Also Associated with Mathematics Research Center, University of Wisconsin, Madison.



and classical results in analytic function theory.

The author's own interest in distributed parameter control theory arose out of consulting experience with Honeywell, Inc., and NASA, starting around 1965 or 1966. In developing the Saturn launch vehicle for the Apollo program, NASA has encountered the problem of transverse vibrations of the booster structure and interaction of those vibrations with liquid sloshing modes in the immense Saturn fuel tanks. While the eventual treatment of that problem was based on finite modal approximations, the problem stimulated a great deal of research aimed at an understanding of the control of vibrations in various distributed parameter settings.

First looking at this problem, under Honeywell-NASA auspices, we thought of modelling the booster structure as an "Euler" beam, the displacement  $w(x, t)$ , which we may take to be scalar here, satisfying

$$\rho(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} (EI(x) \frac{\partial^2 w}{\partial x^2}) = 0 \quad (1.1)$$

along with appropriate boundary conditions including the control inputs, at the longitudinal extremities  $x = 0$ ,  $x = L$ . We got nowhere with our study of this problem initially because the equation (1.1) is not particularly well understood from the mathematical standpoint. There seemed to be no "handles" to grasp. It would not be until the 1969 thesis of Quinn [27] that we would understand how this system works and that it is, in fact, controllable in a rather strong sense.

We knew about the control theory of ordinary differential equations from various papers and from notes and lectures which would later be incorporated into the 1967 treatise on control theory by Lee and Markus [20]. We also knew that hyperbolic partial differential equations in two independent variables reduce to ordinary differential equations satisfied along the characteristics. It was natural, therefore, to look for hyperbolic models which might fit our purpose. Such was provided by the Timoshenko beam equations

$$I_\rho(x) \frac{\partial^2 \psi}{\partial t^2} - k(x) \left( \frac{\partial y}{\partial x} - \psi \right) - \frac{\partial}{\partial x} (EI(x) \frac{\partial \psi}{\partial x}) = 0 \quad (1.2)$$

$$\rho(x) \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x} (k(x) \left( \frac{\partial y}{\partial x} - \psi \right)) = 0, \quad (1.3)$$

which may be viewed as two coupled wave equations. By "wave equations" here, we mean the equation

$$r(x) \frac{\partial^2 z}{\partial t^2} - \frac{\partial}{\partial x} \left( s(x) \frac{\partial z}{\partial x} \right) = 0. \quad (1.4)$$

All coefficient functions shown in (1.2), (1.3), (1.4) are positive on  $0 \leq x \leq L$ . It may be verified that (1.2), (1.3) and (1.4) are hyperbolic in the sense described in [8], [25], e.g.. Since (1.4) is conceptually simpler, it was studied first, with accompanying boundary conditions

$$z(0, t) = 0 \quad (1.5)$$

$$\frac{\partial z}{\partial x}(L, t) = u(t), \quad (1.6)$$

the latter incorporating the control force  $u(t)$ .

While the practical goal in mind was appropriate form of stabilization, we knew that in the case of finite dimensional systems

$$\dot{x} = Ax + Bu$$

an affirmative resolution of the controllability problem, steering from a given  $x(0) = x_0$  to a given  $x(T) = x_1$ , implied the property of stabilization; hence we felt justified in first looking at the state to state controllability problem for (1.4), (1.5), (1.6). The "energy" form for (1.4) is

$$\mathcal{E}(z, \frac{\partial z}{\partial t}) = \frac{1}{2} \int_0^L r(x) \left( \frac{\partial z}{\partial t}(x, t) \right)^2 + s(x) \left( \frac{\partial z}{\partial x}(x, t) \right)^2 dx. \quad (1.7)$$

Given initial and terminal states

$$z(x, 0) = z_0(x), \quad \frac{\partial z}{\partial t}(x, 0) = y_0(x) \quad (1.8)$$

$$z(x, T) = z_1(x), \quad \frac{\partial z}{\partial t}(x, T) = y_1(x) \quad (1.9)$$

of finite energy, i.e.  $\mathcal{E}(z_0, v_0) < \infty$ ,  $\mathcal{E}(z_1, v_1) < \infty$ , we asked if there exists  $u \in L^2[0, T]$  for which the solution of (1.4), (1.5), (1.6) corresponding to the initial state (1.8) assumes the desired terminal state (1.9) at time  $t = T$ . The answer, a qualified "yes", came from two different approaches to the problem. The relationship between these two approaches has, over the years, grown ever more fundamental and has led to a great many very interesting developments. See [34] and [45] in particular.

The first method explored was, as we have already indicated, the method of characteristics. If we let

$$c(x) = \sqrt{\frac{s(x)}{r(x)}} \quad (1.10)$$

and consider families  $X^+$ ,  $X^-$  of "characteristic" curves satisfying

$$\frac{dx^+}{dt} + c(x) = 0, \quad \frac{dx^-}{dt} - c(x) = 0, \quad (1.11)$$

respectively, and then set

$$v^+(x, t) = \frac{\partial z}{\partial t}(x, t) + c(x) \frac{\partial z}{\partial x}(x, t),$$

$$v^-(x, t) = \frac{\partial z}{\partial t}(x, t) - c(x) \frac{\partial z}{\partial x}(x, t),$$

we see readily that on  $X^+ = \{(x^+(t), t)\}$ ,  $X^- = \{(x^-(t), t)\}$ , respectively, we have

$$\frac{d}{dt} v^+(x+(t), t) = c'(x+(t)) \frac{v^+(x+(t), t) - v^-(x+(t), t)}{2} \quad (1.12)$$

$$\frac{d}{dt} v^-(x-(t), t) = c'(x-(t)) \frac{v^+(x-(t), t) - v^-(x-(t), t)}{2} \quad (1.13)$$

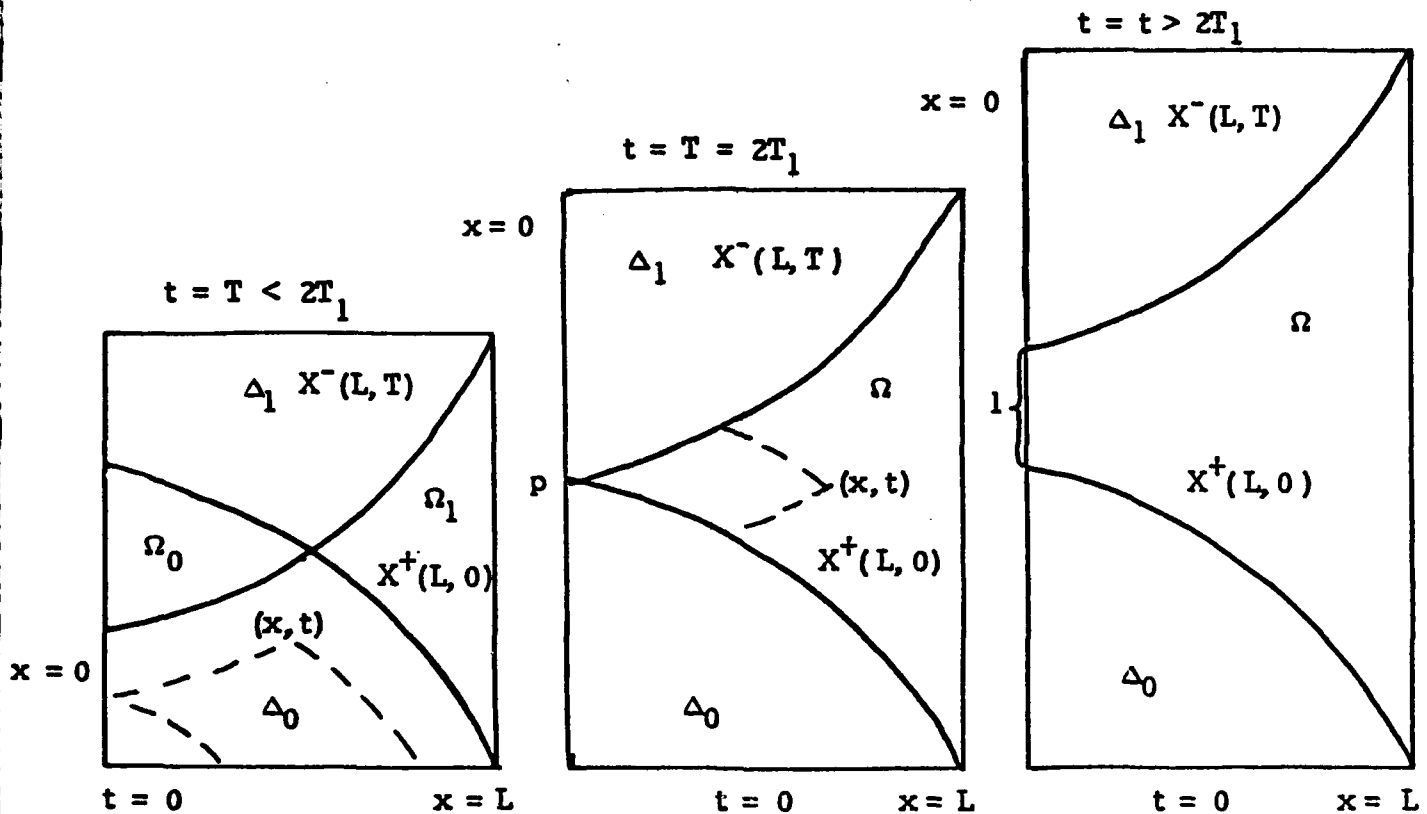


Fig. 1.1: The Method of Characteristics

Because these differential equations are satisfied on different families of characteristics, the coupling between them is more complicated than for the usual system of ordinary differential equations. Nevertheless there is a method of successive approximations, described in [30], [36], which enables solution of these equations in certain regions provided with appropriate boundary data. Such a region is the roughly triangular domain  $\Delta_0$  shown in Fig. 1.1, bounded by  $t = 0$ ,  $x = 0$  and the characteristic  $X^+(L, 0)$ , of the first family described by (1.11), passing through the point  $(L, 0)$ . Together with the boundary data provided by (1.5) and (1.8), it may be seen that the differential equations (1.12), (1.13) determine  $v^+$  and  $v^-$ , and hence  $z(x, t)$ , throughout the domain  $\Delta_0$ . Similarly, these equations together with the data provided by (1.5) and (1.9) determine  $z(x, t)$  in the domain  $\Delta_1$  bounded by  $x = 0$ ,  $t = T$  and the characteristic curve  $X^-(L, T)$ , described by the second equation in (1.11) and passing through the point  $(L, T)$ . Thus the initial and terminal states, described by (1.8), (1.9), together with the boundary condition (1.5) determine  $z(x, t)$  in both  $\Delta_0$  and  $\Delta_1$ .

Whether  $\Delta_0$  and  $\Delta_1$  are disjoint, or have a region,  $\Omega_0$ , of overlap, depends on the time  $T$  allotted for control. The time required for the curve  $X^+(0, L)$  to pass from  $X = L$  to  $x = 0$  is

$$T_1 = \int_0^L \frac{dx}{c(x)} \quad (1.14)$$

and this is also the time required for  $X^-(L, T)$  to pass from  $x = 0$  to  $x = L$ . We summarize the control situation, depending on the relationship between  $T$  and  $T_1$ .

Case  $T < 2T_1$ . Here  $\Delta_0$  and  $\Delta_1$  overlap and the determinations of  $z(x, t)$  in the overlap region  $\Omega_0 = \Delta_0 \cup \Delta_1$  provided by (1.8) and (1.9) need not and, in general, will not agree. There can, in such cases of disagreement, be no solution of (1.4), equivalently (1.12), (1.13), in the region  $R_T = \{(x, t) | 0 \leq x \leq L, 0 \leq t \leq T\}$ . The control function  $u(t)$ , shown in (1.6), never enters the picture because it cannot affect the solution of (1.4) in  $\Delta_0$  or  $\Delta_1$  if (1.8), (1.9) are satisfied at  $t = 0$ ,  $t = T$ , respectively.

Case  $T = 2T_1$ . Here the two "domains of determinacy",  $\Delta_0$  and  $\Delta_1$ , just fail to overlap; their boundaries have exactly one point in common,  $t = T_1$ ,  $x = 0$ . The initial and terminal conditions (1.8) and (1.9) determine  $z(x, t)$  in  $\Delta_0$  and  $\Delta_1$ , respectively. Another process of integration of the coupled

differential equations (1.12) and (1.13) permits unique extension of  $z(x, t)$ , equivalently  $v^+(x, t)$ ,  $v^-(x, t)$ , into the domain  $\Omega$ . The control steering (1.8) to (1.9) is then uniquely determined from this extension and (1.6).

The determinations of  $z(x, t)$  in  $\Delta_0$  and  $\Delta_1$  may fail to match smoothly at the point  $p: x = 0, t = T_1$ . This results in discontinuities of  $v^+$  along  $X^+(L, 0)$  and of  $v^-$  along  $X^-(L, T)$  in general.

Case  $T > 2T_1$ . The only difference between this case and the case  $T = 2T_1$  lies in the line segment  $l: x = 0, T_1 < t < T - T_1$ , which replaces the point  $p$  of the case  $T = 2T_1$ . Extension of  $z(x, t)$  from  $\Delta_0 \cup \Delta_1$  into  $\Omega$  cannot be carried out until the boundary condition (1.5), which yields  $\partial z / \partial t(0, t) = 0$ , is augmented by arbitrary data

$$\frac{\partial z}{\partial x}(0, t) = \zeta(t), \quad (0, t) \in l. \quad (1.14)$$

Once this is done, extension of  $z(x, t)$  into  $\Omega$  proceeds much as before. (See [30], [36] for details of the extension process.) The arbitrary function  $\zeta(t)$  can be designed so as to eliminate discontinuities of the solution along  $X^+(L, 0)$  and  $X^-(L, T)$ , to satisfy some criterion of optimality (see [30] e.g.) or to fulfill any other appropriate design objective.

If the partial differential equation (1.4) is combined with boundary conditions different from (1.5), (1.6), but still admissible for (1.4), the cases  $T < 2T_1$ ,  $T > 2T_1$  remain as above. The rather delicate situation at  $T = 2T_1$  depends on the specific form of the boundary conditions. For example, the boundary conditions

$$z(0, t) = 0, \quad z(L, t) = u(t)$$

lead, in case  $T = 2T_1$ , to a situation where the desired control is not unique; it has the form

$$u(t) = \hat{u}(t) + \gamma \tilde{u}(t)$$

where  $\tilde{u}(t)$  is a non-zero control steering the zero initial state into the zero final state and  $\gamma$  is an arbitrary constant. By contrast, the boundary conditions

$$\frac{\partial z}{\partial x}(0, t) = 0, \quad \frac{\partial z}{\partial x}(l, t) = u(t) \quad (1.16)$$

lead, in case  $T = 2T_1$ , to a situation where the desired control  $u(t)$  does not, in general, exist. (See [31], [37] for more details.)

The analysis of more complicated systems of hyperbolic equations, such

as the Timoshenko system (1.2), (1.3), is in general rather complicated but there are some special cases, including appropriate boundary conditions, for which the analysis is fairly simple. In [M] a discussion is given permitting analysis of the free boundary case

$$\frac{\partial \psi}{\partial x}(0, t) = 0, \quad \psi(0, t) - \frac{\partial y}{\partial x}(0, t) = 0, \quad (1.17)$$

$$\frac{\partial \psi}{\partial x}(L, t) = u_1(t), \quad \psi(L, t) - \frac{\partial y}{\partial x}(L, t) = u_2(t). \quad (1.18)$$

It may be shown that all cross-coupling is of low order and the problem is essentially equivalent to two problems (1.4) with boundary conditions (1.16). Two critical times are involved. With (cf. (1.14))

$$c_1(x) = \left( \frac{EI(x)}{I\rho(x)} \right)^{\frac{1}{2}}, \quad T_1 = \int_0^L \frac{dx}{c_1(x)}, \quad (1.19)$$

$$c_2(x) = \left( \frac{k(x)}{\rho(x)} \right)^{\frac{1}{2}}, \quad T_2 = \int_0^L \frac{dx}{c_2(x)}, \quad (1.20)$$

it may be shown that finite energy states are controllable if and only if

$$T \geq 2 \max\{T_1, T_2\}.$$

The essential details of the analysis are given in [30] and are quite similar to what we have briefly outlined here for (1.4), (1.5), (1.6).

It is immediately clear that the method of characteristics is specially adapted to controls  $u(t)$  acting at a point, as in (1.6). This is true because the control determination occurs at the very last stage of the analysis, after the controlled solution has been computed. If the control  $u(t)$ , itself scalar, acts on the system through a "control distribution function"  $g(x)$ , as in (cf. (1.4))

$$r(x) \frac{\partial^2 z}{\partial t^2} - \frac{\partial}{\partial x} \left( s(x) \frac{\partial z}{\partial x} \right) = g(x) u(t), \quad (1.21)$$

homogeneous boundary conditions (cf. (1.5), (1.6))

$$z(0, t) = 0, \quad \frac{\partial z}{\partial x}(L, t) = 0 \quad (1.22)$$

applying at the boundaries, we face what appears at first glance to be a rather different situation than what obtains in (1.4), (1.5), (1.6), for even the equations corresponding to (1.12), (1.13) will involve the unknown control  $u(t)$  in this situation; one cannot proceed by filling out  $z(x, t)$  in successive domains as before; a completely different approach is required. Such an approach can be found in the study of moment problems - a technique developed

by several authors (see [3], [12], [15], [14], [23]). The technique has the advantage, from the point of view of approximation of being intimately connected with the modal representation of the system based on the natural modes of vibration, or eigenfunctions of the operator  $-r(x)^{-1}(\partial/\partial x)(s(x)(\partial z/\partial x))$ .

It is known (see [1], [7]) that the operator

$$Lz = -r(x)^{-1} \frac{\partial}{\partial x} (s(x) \frac{\partial z}{\partial x}) \quad (1.23)$$

with boundary conditions conformable with (1.22) has eigenvalues

$$\lambda_k = \left(\frac{2k-1}{2}\right)^2 \frac{\pi^2}{T_1^2} + \varepsilon_k, \quad k = 1, 2, 3, \dots \quad (1.24)$$

where the  $\varepsilon_k$  are uniformly bounded and  $T_1$  is related to  $c(x)$  by (1.14). The corresponding eigenfunctions,  $\varphi_k(x)$ ,  $k = 1, 2, 3, \dots$ , form an orthonormal basis for  $L_r^2[0, L]$  (which consists of the same functions as  $L^2[0, L]$  but has the inner product

$$(\varphi, \psi)_r = \int_0^L r(x) \varphi(x) \psi(x) dx. \quad (1.25)$$

Every finite energy solution  $z(x, t)$  of (1.21), (1.22), i.e. every solution for which the integral (1.7) is bounded for all  $t$ , can be expanded in the form

$$\ddot{z}(x, t) = \sum_{k=1}^{\infty} z_k(t) \varphi_k(x)$$

where, if we assume the control distribution function  $f(x)$  has the expression

$$g(x) = \sum_{k=1}^{\infty} g_k \varphi_k(x)$$

convergent in  $L_r^2[0, L]$ , the  $z_k(t)$  satisfy

$$z_k'' + \lambda_k z_k = g_k u(t), \quad k = 1, 2, \dots \quad (1.26)$$

Letting

$$\omega_k = \lambda_k^{\frac{1}{2}}, \quad k = 1, 2, 3, \dots$$

and using the transformation

$$\begin{pmatrix} z_k \\ \dot{z}_k \end{pmatrix} = \begin{pmatrix} 1/i\omega_k & -1/i\omega_k \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \quad (1.27)$$

one arrives at the system

$$\begin{pmatrix} \dot{\eta}_k \\ \dot{\zeta}_k \end{pmatrix} = \begin{pmatrix} i\omega_k & 0 \\ 0 & -i\omega_k \end{pmatrix} \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} + \begin{pmatrix} g_k/2 \\ g_k/2 \end{pmatrix} u(t). \quad (1.28)$$

It may be seen that finite energy states are those for which

$$\sum_{k=1}^{\infty} [(\dot{z}_k)^2 + \lambda_k (z_k)^2] < \infty$$

and this becomes, in terms of  $\eta_k, \zeta_k$ ,

$$\sum_{k=1}^{\infty} (|\eta_k|^2 + |\zeta_k|^2) < \infty. \quad (1.29)$$

Integrating (1.28), we have, for  $T > 0$ ,

$$\begin{aligned} \eta_k(T) - e^{i\omega_k T} \eta_k(0) &= \frac{g_k}{2} \int_0^T e^{i\omega_k(T-t)} u(t) dt \\ \zeta_k(T) - e^{i\omega_k T} \zeta_k(0) &= \frac{g_k}{2} \int_0^T e^{-i\omega_k(T-t)} u(t) dt. \end{aligned}$$

Assuming the controllability condition

$$g_k \neq 0, \quad k = 1, 2, 3, \dots$$

we see that the problem of steering between the given states at times 0 and T reduces to the moment problem

$$\int_0^T e^{i\omega_k s} f(s) ds = \frac{\alpha_k}{g_k}, \quad k = 1, 2, 3, \dots, \quad (1.30)$$

$$\int_0^T e^{-i\omega_k s} f(s) ds = \frac{\beta_k}{g_k}, \quad k = 1, 2, 3, \dots, \quad (1.31)$$

where  $s = T-t$ ,  $f(s) = u(T-s)$ , and

$$\alpha_k = 2(\eta_k(T) - e^{i\omega_k T} \eta_k(0)), \quad \beta_k = 2(\zeta_k(T) - e^{-i\omega_k T} \zeta_k(0)) \quad (1.32)$$

are square summable.

To solve the moment problem we resort to the theory of nonharmonic Fourier series as developed by Paley and Wiener [24], Levinson [21], Schwartz [42] and many others. (An excellent expository treatment [49] by R. Young has recently appeared.) The following is known; the three cases being divided in a manner conformable with the three cases discussed earlier.

Case  $T < 2T_1$ . The functions  $e^{i\omega_k t}$ ,  $e^{-i\omega_k t}$ ,  $k = 1, 2, 3, \dots$ , are linearly dependent in  $L^2[0, T]$  in a rather strong sense. Any one of these functions, indeed, any finite number of them, lie in the closed span of the remaining functions (which, in fact, is equal to the whole space  $L^2[0, T]$ ). As a result the moment problem (1.30), (1.31) cannot, in general, be solved.



Suppose, e.g., all but finitely many of the  $\alpha_k, \beta_k$ , say  $k = K+1, K+2, \dots$ , were equal to zero, while some of the  $\alpha_k, \beta_k$ ,  $k = 1, 2, \dots, K$ , are non-zero. The linear dependence just referred to shows that such a problem can have no solution; the equations

$$\int_0^T e^{i\omega_k s} f(s) ds = \int_0^T e^{-i\omega_k s} f(s) ds = 0, \quad k > K$$

imply that the same equations must hold for  $k \leq K$ .

Case T = T<sub>1</sub>. Here the functions  $e^{i\omega_k s}, e^{-i\omega_k s}$ ,  $k = 1, 2, 3, \dots$  form a Riesz basis for  $L^2[0, 2T_1]$ . Every function  $h \in [0, 2T_1]$  has the unique convergent expansion

$$h(s) = \sum_{k=1}^{\infty} [h_k e^{i\omega_k s} + h_{-k} e^{-i\omega_k s}] \quad (1.33)$$

and there are positive numbers  $c, C$ , such that

$$c^{-2} \|h\|_{L^2[0, 2T]}^2 \leq \sum_{k=1}^{\infty} (|h_k|^2 + |h_{-k}|^2) \leq C^2 \|h\|_{L^2[0, 2T]}^2 \quad (1.34)$$

Further, there is a unique dual basis of biorthogonal elements  $p_k, p_{-k} \in L^2[0, 2T_1]$  such that

$$\int_0^{2T_1} e^{i\omega_k s} p_l(s) ds = \delta_{k,l}, \quad \begin{matrix} k = \pm 1, \pm 2, \dots \\ l = \pm 1, \pm 2, \dots \end{matrix} \quad (1.35)$$

$$\delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

which engenders expansions similar to (1.33), the roles of  $c, C$  in the inequalities paralleling (1.34) being reversed. The formal solution of (1.30), (1.31) is then uniquely given by

$$f(s) = \sum_{k=1}^{\infty} \frac{1}{g_k} [\alpha_k p_k(s) + \beta_k p_{-k}(s)]. \quad (1.36)$$

If we have

$$\lim_{k \rightarrow \infty} |g_k| = 0, \quad (1.37)$$

as would be the case, e.g., if  $g \in L^2[0, L]$ , then the conditions for convergence of (1.36) are more stringent than just the square summability of the  $\alpha_k, \beta_k$  given by (1.32). We need

$$\sum_{k=1}^{\infty} (|\frac{\alpha_k}{g_k}|^2 + |\frac{\beta_k}{g_k}|^2) < \infty.$$

As a consequence we can steer (1.21) from any finite energy initial state to a dense (in the energy norm) subspace of final states, or vice versa, but we cannot steer between arbitrary finite energy states during  $[0, 2T_1]$  if (1.37) is true. The case of boundary control (1.6), already treated by the method of characteristics, will be discussed more extensively below. In that case the coefficients  $g_k$  in (1.26) are bounded and bounded below. The result is, in that situation, that we obtain the same result this way as by the method of characteristics - given finite energy initial and terminal states, there is a unique control  $u \in L^2[0, 2T_1]$  steering the one to the other.

Case  $T > 2T_1$ . The main difference between this case and the preceding is that here the functions  $e^{i\omega_k s}$ ,  $e^{-i\omega_k s}$ ,  $k = 1, 2, 3, \dots$ , form a Riesz basis for a proper subspace,  $E$ , of  $L^2[0, T]$ . The biorthogonal functions  $p_k$ ,  $p_{-k}$  exist, but are unique only if we require that they lie in  $E$  - or we impose some comparable condition. If we agree that  $\tilde{p}_k$ ,  $\tilde{p}_{-k}$  belong to  $E$ , then any elements

$$p_k = \tilde{p}_k + q_k, \quad p_{-k} = \tilde{p}_{-k} + q_{-k}$$

with  $q_k, q_{-k} \in E^\perp \subset L^2[0, T]$  still form a biorthogonal set relative to the  $e^{i\omega_k s}$ ,  $e^{-i\omega_k s}$ . The convergence properties of series involving the  $\tilde{p}_k$ ,  $\tilde{p}_{-k}$  are much the same as in the preceding case. As a result we have the same control capability as in the case  $T = 2T_1$  but controls are not unique. Indeed, if  $\tilde{u}$  is a control steering between two given states, the family of controls  $\tilde{u} + \hat{u}$ ,  $\hat{u} \in E^\perp$ , all realize the same control objective. Again, this non-uniqueness should be compared with the similar property observed for  $T > 2T_1$  in applying the method of characteristics.

Using the theory of distributions and related material, boundary value control situations such as (1.6) can be included in the same framework as (1.26) but with  $g$  in a larger space than  $L^2[0, L]$ ;  $g$  should be a linear functional (in general unbounded on  $L^2[0, L]$ ) whose domain includes the domain,  $\mathcal{D}(L)$ , of the self adjoint operator  $L$ , given by (1.23), with the given homogeneous boundary conditions. The  $g_k$  are the values which  $g$  assumes at the eigenfunctions  $\varphi_k \in \mathcal{D}(L)$ . A detailed study of these "admissible input elements" is provided in [17]. In this way a unification of the boundary and distributed control cases may be achieved. One consequence of this is that the biorthogonal functions  $p_k$ ,  $p_{-k}$  which play such an

important role in the method based on the moment problem (1.30), (1.31) can actually be obtained through the more constructive method of characteristics as controls steering from a zero initial state (say) to final states constructed using a single eigenfunction  $\varphi_k$  of  $L$ .

We began our discussion here with the Euler beam equation (1.1). For definiteness, let us add a distributed control term (scalar input) and specific boundary conditions so that we have

$$\rho(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} (EI(x) \frac{\partial^2 w}{\partial x^2}) = g(x) u(t) \quad (1.38)$$

$$\frac{\partial^2 w}{\partial x^2} (0, t) = \frac{\partial^3 w}{\partial x^3} (0, t) = 0, \quad (1.39)$$

$$\frac{\partial^2 w}{\partial x^2} (L, t) = \frac{\partial^3 w}{\partial x^3} (L, t) = 0. \quad (1.40)$$

In 1969 J.P. Quinn, in his doctoral thesis [27], studied the controllability properties of a class of systems including this one. Here the operator

$$Aw = \frac{1}{\rho(x)} \frac{\partial^2}{\partial x^2} (EI(x) \frac{\partial^2 w}{\partial x^2})$$

on the domain in  $H^4[0, L]$  consisting of functions obeying boundary conditions conformable with (1.39), (1.40) has eigenfunctions  $\varphi_k(x)$  forming an orthonormal basis for  $L^2[0, L]$  and the corresponding eigenvalues  $\lambda_k$  grow like  $k^4$  as  $k \rightarrow \infty$ . With  $\omega_k = \lambda_k^{\frac{1}{2}}$  we obtain a system similar to (1.28), using a transformation like (1.27) applied to the second order differential equations resulting from the eigenfunction decomposition:

$$\ddot{w}_k + \lambda_k w_k = g_k u(t), \quad k = 0, 1, 2, 3, \dots$$

(a slight modification of (1.27), (1.28) is necessary for  $\lambda_0 = 0$ ; see [34]). Again there results

$$\begin{pmatrix} \dot{\eta}_k \\ \dot{\zeta}_k \end{pmatrix} = \begin{pmatrix} i\omega_k & 0 \\ 0 & -i\omega_k \end{pmatrix} \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} + \begin{pmatrix} \frac{g_k}{2} \\ \frac{g_k}{2} \end{pmatrix} u(t) \quad (1.41)$$

and the energy expression for (1.38) is, equivalently,

$$\frac{1}{2} \int_0^L \left[ \rho(x) \left( \frac{\partial w}{\partial t} \right)^2 + EI(x) \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx$$

or

$$\frac{1}{2} \sum_{k=0}^{\infty} [(\dot{w}_k)^2 + \lambda_k (w_k)^2]$$

or

$$\frac{1}{2} \sum_{k=0}^{\infty} |\eta_k|^2 + |\zeta_k|^2,$$

all  $< \infty$  for "finite energy" states.

Quinn was able to show in this case that the functions  $e^{i\omega_k s}, e^{-i\omega_k s}$ ,  $k = 0, 1, 2, 3, \dots$  (for  $k = 0$  replace  $e^{i\omega_k s}, e^{-i\omega_k s}$  by  $1, s$ ) are linearly independent in  $L^2[0, T]$  for every  $T > 0$  (this result by itself had already been obtained much earlier by Ingham [18] who shows, in effect, that these functions form a Riesz basis for a closed subspace of  $L^2[0, T]$  for every  $T > 0$ ) and, additionally, that there is a positive number,  $M(T)$ , such that if the (non-unique) biorthogonal functions  $p_k(s), p_{-k}(s)$  are appropriately selected in  $L^2[0, T]$ , these functions are continuous and satisfy the pointwise bounds

$$|p_k(s)| \leq M(T), \quad |p_{-k}(s)| \leq M(T), \quad s \in [0, T]. \quad (1.42)$$

The fact that the  $e^{i\omega_k s}, e^{-i\omega_k s}$  form a Riesz basis for a closed subspace of  $L^2[0, T]$ ,  $T > 0$ , implies that initial states and terminal states with (in terms of (1.41)) expansion coefficients  $\eta_{k,0}, \zeta_{k,0}$  and  $\eta_{k,1}, \zeta_{k,1}$  can be steered, one to the other, during  $[0, T]$  with  $u \in L^2[0, T]$ , provided that

$$\sum_{k=0}^{\infty} (|\frac{\eta_{k,0}}{g_k}|^2 + |\frac{\zeta_{k,0}}{g_k}|^2) < \infty, \quad \sum_{k=0}^{\infty} (|\frac{\eta_{k,1}}{g_k}|^2 + |\frac{\zeta_{k,1}}{g_k}|^2) < \infty.$$

The boundedness property (1.42) shows we can also control states for which

$$\sum_{k=0}^{\infty} (|\frac{\eta_{k,0}}{g_k}| + |\frac{\zeta_{k,0}}{g_k}|) < \infty,$$

$$\sum_{k=0}^{\infty} (|\frac{\eta_{k,1}}{g_k}| + |\frac{\zeta_{k,1}}{g_k}|) < \infty,$$

this being possible with a control function  $u(t)$  uniformly bounded and continuous on  $[0, T]$ .

We have noted in connection with the Timoshenko beam system (1.2), (1.13)

(1.17), (1.18), that an adequate control theory, based on the method of characteristics, exists when we have two separate control functions,  $u_1(t)$  and  $u_2(t)$ , with which to control the lateral deflection and shear deformation separately. An open question is the adequacy of control using a single control input, so that (1.17), (1.18) becomes, e.g.,

$$\frac{\partial \psi}{\partial x}(L, t) = \alpha u(t), \quad \psi(L, t) - \frac{\partial y}{\partial x}(L, t) = \beta u(t)$$

with  $\alpha^2 + \beta^2 \neq 0$ . This problem is a special case of the more general question of the controllability of linear hyperbolic systems of dimension  $n = 2m$ , involving  $m$  pairs of characteristics, each pair describing a given wave mode propagating in two opposite directions, by means of fewer than  $m$  control inputs. Some work has been done in this direction by R. G. Teglus in his thesis [45] and by N. Wick [47], but it is safe to say that no very general criteria for this problem have yet appeared. Particularly valuable, it seems to this author, would be a study of the Timoshenko beam system from the singular perturbation standpoint, elucidating the behavior of solutions and controllability properties as the modulus of elasticity in shear,  $k(x)$  in (1.2), (1.3), tends to infinity.

#### STABILIZATION, CANONICAL FORMS, EIGENVALUE PLACEMENT, etc.

As all practicing engineers will know, controllability in itself is rarely the prime goal of control system design. Stability, and related criteria such as robustness, insensitivity to particular input frequency bands, etc., are more commonly uppermost in mind. Additionally, there is the question of state estimation from lower dimensional, noisy observations in order to implement linear feedback control policies. These subjects have been pursued almost ad nauseum for linear, finite dimensional systems. In the case of distributed parameter systems, and hyperbolic systems in particular, the literature on this subject remains rather sparse and spotty in its coverage.

As in the case of linear finite dimensional systems, stability and stabilization studies for linear partial differential equations have tended to cluster around two dominant approaches: the Liapounov approach, primarily carried out in connection with systems involving some form of "conservation of energy" law, and the spectral approach, determining if, or making certain that, the

eigenvalues of the system lie in an appropriate subset of the left half plane. The spectral approach suffers from the disadvantages of greater intricacy of computation and the need to show that the spectrum location does, in fact, determine the asymptotic behavior of the system. The latter brings in questions of completeness and linear independence of the eigenvectors of the system.

We will begin with a short discussion of what has been done with Liapounov methods. On the theoretical side one can start with a system

$$\dot{x} = Cx, \quad (2.1)$$

$C$  generating a strongly continuous semigroup  $S(t)$  in the Hilbert space  $X$  (we may have started with a control system  $\dot{x} = Ax + Bu$ , set  $u = Kx$ , then  $C = A + BK$ ). We set up a quadratic functional

$$V(x) = (x, Qx),$$

where  $Q$  is a bounded, positive, self adjoint operator on  $X$  with  $Q \geq qI$  for some  $q > 0$ , to serve as a Liapounov function. One may then show that for  $t_2 > t_1$  and  $x(t) = S(t)x_0$  a "solution" of (2.1), that

$$\begin{aligned} & (x(t_2), Qx(t_2)) - (x(t_1), Qx(t_1)) \\ & = - \int_{t_1}^{t_2} (x(s), Wx(s)) ds \end{aligned}$$

for some positive self adjoint operator  $W$  so that, in some sense which one needs to make precise in individual cases,

$$C^*Q + QC + W = 0, \quad (2.2)$$

the Liapounov operator equation, is satisfied. An important result, due to Datko [10], states that if

$$\int_0^\infty (x(t), Qx(t)) dt < \infty \quad (2.3)$$

for every initial state  $x_0 \in X$ , then the semigroup  $S(t)$  is exponentially damped, i.e.

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for positive numbers  $M, \gamma$ . The condition (2.3) is satisfied if  $W \geq wI$  for some  $w > 0$ , as may easily be verified.

Consider the linear symmetric hyperbolic system in  $L^2_n[0, L] = (L^2[0, L])^n$

$$E(x) \frac{\partial w}{\partial t} = A(x) \frac{\partial w}{\partial x} + B(x)w + f(x, t) \quad (2.4)$$

where  $E(x)$ ,  $A(x)$ ,  $B(x)$  are continuously differentiable  $m \times m$  matrices defined for  $x \in [0, L]$ ,  $E(x)$  symmetric and positive definite,  $A(x)$

symmetric. The wave and Timoshenko equations can be written in this form. The "energy" usually is expressed as

$$e(t) = \frac{1}{2} \int_0^L (w(x, t), E(x) w(x, t)) dx.$$

With appropriately "conservative" or "dissipative" boundary conditions at  $x = 0$ ,  $x = L$ , one finds that for  $t_2 > t_1$

$$e(t_2) - e(t_1) = \int_{t_1}^{t_2} \left\{ \frac{1}{2} \int_0^L (w(x, t), [B(x) + B(x)^* - \dot{A}(x)] w(x, t)) dx + \int_0^L (w(x, t), f(x, t)) dx \right\} dt.$$

If  $B(x) + B(x)^* - \dot{A}(x)$  is uniformly negative definite or if the  $n$  dimensional control function  $f(x, t)$  may be arbitrarily specified as a function of  $x$  and  $t$ , one may use feedback

$$f(x, t) = K(x) w(x, t) \quad (2.5)$$

in such a way that

$$e(t_2) - e(t_1) = - \int_{t_1}^{t_2} \int_0^L (w(x, t), W(x) w(x, t)) dx dt \quad (2.6)$$

with  $W(x)$  uniformly positive definite and symmetric on  $[0, L]$ . Then one can apply Datko's result, or more simple arguments, to show that solutions of (2.3), (2.4) are uniformly exponentially damped in  $L_n^2[0, L]$  norm.

Note, however, that if  $B(x) + B(x)^* - \dot{A}(x) = 0$  or for some other reason fails to be positive definite, and if

$$f(x, t) = D(x) u(x, t) \quad \text{or} \quad f(x, t) = D(x) u(t)$$

with  $\dim u(x, t) = r < R$  in the first instance,  $u$  a function of  $t$  only in the second instance, then we cannot, in general, achieve (2.5) with  $W(x)$  uniformly positive definite. Comparable difficulties arise when boundary control is employed. In such cases it is a form of the La Salle "invariance principle" (see, e.g. [19]) which must be appealed to, rather than the basic Liapounov theory, for an analysis of presumed asymptotic stability properties of the system. This has been discussed in some detail in [36] and [33] and we give only the briefest outline here.

The "invariance principle", as it applies to finite dimensional systems, relies heavily on the compactness of the " $\omega$ -limit set" of the system in order to reach the final conclusion of asymptotic stability. Comparable compactness properties associated with the solutions of an infinite dimensional system are

generally difficult to realize but the initial attempts to extend the theory nonetheless relied on establishing some sort of compactness property. One of the first contributions in this direction was due to Dafermos [9] who studied weak damping of the wave equation, relying on the almost periodic nature of the system solutions to provide the required compactness. Slemrod [43] studied the boundary damped wave equation by introducing suitably weakened topologies — as compared with the usual topology associated with the energy norm — and was able to conclude a correspondingly weakened form of asymptotic stability.

Knowing that controllability implies stabilizability in the case of autonomous finite dimensional linear systems, we are not surprised to find controllability playing a role in the study of asymptotic stability and stabilization properties of autonomous infinite dimensional linear systems. This is discussed in some detail in the paper [28] by J. P. Quinn and the author and also in [33]. Systems of the form (2.3), but with the control appearing in the boundary conditions, are studied in [28] prior to the main discussion on the boundary damped, higher dimensional wave equation. We can give an idea of the flavor of the arguments employed using a simple example based on the wave equation (1.4) with control appearing in the boundary conditions (1.5), (1.6). If in this system one employs the feedback law

$$u(t) = -\gamma \frac{\partial w}{\partial t}(L, t), \quad (2.7)$$

the closed loop system is (1.4), (1.5) together with the "closed-loop" Robin type boundary condition

$$\frac{\partial w}{\partial x}(L, t) + \gamma \frac{\partial w}{\partial t}(L, t) = 0. \quad (2.8)$$

Here a short computation shows that with the energy  $e(t)$  defined by the expression (1.7) we have, for  $t_2 > t_1$ ,

$$e(t_2) - e(t_1) = -\gamma p(L) \int_{t_1}^{t_2} \frac{\partial w}{\partial t}(L, t)^2 dt. \quad (2.9)$$

It is not feasible to fit this situation into the general pattern based on the Liapounov operator equation (2.2) but, since we expect (correctly) that, along with (1.4), (1.5), (2.8)

$$\frac{\partial w}{\partial t}(L, t) \equiv 0 \Rightarrow w(x, t) \equiv 0,$$

an "invariance principle" type of argument appears to be in order. But we will use a variation on this procedure which makes use of the controllability already established in Section 1. Let  $v(x, t)$  be a controlled solution of



(1.4), (1.5), (1.6),  $u(t)$  being selected so as to steer the initial state

$$v(x, 0) = w(x, 0), \quad \frac{\partial v}{\partial t}(x, 0) = \frac{\partial w}{\partial t}(x, 0), \quad (2.10)$$

agreeing with the initial state of the solution  $w(x, t)$  of (1.4), (1.5), (2.8), to the zero final state

$$v(x, T_1) = 0, \quad \frac{\partial v}{\partial t}(x, T_1) = 0, \quad (2.11)$$

$T_1$  as described earlier. Defining the "energy inner product"

$$\langle w(\cdot, t), v(\cdot, t) \rangle = \int_0^L \left[ p(x) \frac{\partial w}{\partial t}(x, t) \frac{\partial v}{\partial t}(x, t) + p(x) \frac{\partial w}{\partial x}(x, t) \frac{\partial v}{\partial x}(x, t) \right] dx$$

it is found, using (1.5), (2.8), (2.10), (2.11), that

$$\begin{aligned} \langle w(\cdot, 0), v(\cdot, 0) \rangle - \langle w(\cdot, 2T_1), v(\cdot, 2T_1) \rangle &= \|w(\cdot, 0)\|_e^2 = \\ &= -p(L) \int_0^{2T_1} \left[ \frac{\partial w}{\partial x}(L, t) \frac{\partial v}{\partial t}(L, t) + \frac{\partial v}{\partial x}(L, t) \frac{\partial w}{\partial t}(L, t) \right] dt \\ &= p(L) \int_0^{2T_1} \frac{\partial w}{\partial t} \left[ \gamma \frac{\partial v}{\partial t}(L, t) + \frac{\partial v}{\partial x}(L, t) \right] dt \\ &= p(L) \int_0^{2T_1} \frac{\partial w}{\partial t}(L, t) \left[ \gamma \frac{\partial v}{\partial t}(L, t) + u(t) \right] dt. \end{aligned}$$

Here  $\|w(\cdot, 0)\|_e^2$ , the energy norm at  $t = 0$ , is  $2e(0)$ . Applying the Schwartz inequality

$$\begin{aligned} 4e(0)^2 &\leq p(L) \int_0^{2T_1} \frac{\partial w}{\partial t}(L, t)^2 dt \\ &\quad \cdot \int_0^{2T_1} (\gamma \frac{\partial v}{\partial t}(L, t) + u(t))^2 dt. \end{aligned} \quad (2.12)$$

A slightly more detailed study of the control problem for (1.4), (1.5), (1.6) in the case  $T = 2T_1$  (or  $T > 2T_1$ ) shows that control from an initial state  $w(x, 0)$ ,  $\frac{\partial w}{\partial t}(x, 0)$  to 0, 0 at time  $2T_1$  is realized with a control  $u(t)$  which satisfies

$$\int_0^{2T_1} u(t)^2 dt \leq K_0 e(0)$$

and, for the resulting controlled solution we have

$$\int_0^{2T_1} \frac{\partial v}{\partial t}(L, t)^2 dt \leq K_1 e(0)$$

for certain positive constants  $K_0$  and  $K_1$ . Then (2.12) easily yields

$$\gamma p(L) \int_0^{2T_1} \frac{\partial w}{\partial t}(L, t)^2 dt \geq \frac{e(0)^2}{2(K_0 + \gamma^2 K_1) e(0)} \equiv Ke(0)$$

and, setting  $t_1 = 0$ ,  $t_2 = 2T_1$  in (2.9), we have

$$e(2T_1) \leq e(0) - K e(0) = (1 - K) e(0). \quad (2.13)$$

Since  $e(2T_1)$  is, from (2.9), (2.13), positive and less than or equal to  $e(0)$  we conclude  $0 < 1 - K < 1$ .

Repeating the above argument on successive intervals  $[0, 2T_1]$ ,  $[2T_1, 4T_1]$ ,  $\dots$ ,  $[2kT_1, 2(k+1)T_1]$ ,  $\dots$  and using the monotonicity of  $e(t)$ , as implied by (2.9), we conclude that  $e(t)$  decays exponentially to 0 as  $t \rightarrow \infty$ . The same general argument can be used with a fairly wide class of boundary damped linear symmetric hyperbolic systems (2.4) and with many other systems which are energy conserving in the uncontrolled situation and suitably strong controllability properties. The Timoshenko system (1.2), (1.3), with appropriate boundary conditions, is in this class. As far as the author is aware, the Euler beam model (1.1) has not yet been studied from this point of view.

The spectral approach, as we have already indicated, involves a direct analysis of the eigenvalues and eigenfunctions or, more generally, the spectrum and invariant subspaces, of the generating operator  $C$  for a given system  $\dot{x} = Cx$ , possibly derived from a control system  $\dot{x} = Ax + Bu$  by the use of linear feedback  $u = Kx$  so that  $C = A + BK$ . A fairly common case, which can be treated with minimal difficulty, arises when all but finitely many of the eigenvalues of  $C$  have negative real parts. Under generically valid controllability-type conditions it is then possible to move the unstable eigenvalues into the left half plane while either keeping the stable eigenvalues fixed or else maintaining a certain margin of stability. Work of this sort has been carried out by Triggiani [46], Sakawa [40], [41] and others.

A somewhat more challenging task arises when one starts with a system having infinitely many eigenvalues in the closed right half plane (usually one considers a conservative system wherein all of the eigenvalues of  $C$  are purely imaginary) and one attempts to devise a feedback law to move all of these eigenvalues over into the open left half plane. A number of procedures have been examined in this connection.

In [32] a second order system with scalar control

$$\ddot{x} + Ax = bu, \quad x, b \in X, \quad (2.14)$$

is studied,  $X$  being a real Hilbert space and  $A$  an unbounded positive self adjoint operator on  $X$ . Assuming that  $A$  has a Riesz basis of eigenvectors  $\phi_k$ ,  $k = 1, 2, 3, \dots$ , in  $X$ , and corresponding positive eigenvalues  $\lambda_k$

increasing with  $k$ ,  $k = 1, 2, 3, \dots$ ,  $x$  and  $b$  may be expanded as

$$x = \sum_{k=1}^{\infty} x_k \phi_k, \quad b = \sum_{k=1}^{\infty} b_k \phi_k, \quad (2.15)$$

convergent in  $X$ , with square summable coefficients. We assume the minimal condition for approximate controllability

$$b_k \neq 0, \quad k = 1, 2, 3, \dots$$

The energy form is  $\frac{1}{2}[(\dot{x}, \dot{x}) + (x, Ax)] = e$  and elementary computations show that for (2.14) and for any  $T > 0$

$$e(T) - e(0) = \int_0^T (\dot{x}(t), b) u(t) dt. \quad (2.16)$$

It follows that with

$$u(t) = -\gamma(\dot{x}(t), b) \quad (2.17)$$

the energy  $e(t)$  is non-increasing with increasing  $t$ . So far this is basically a Liapounov approach employing what is known in the engineering literature as an ILAF (Identical Location of Accelerometer and Forces) approach. The resulting closed loop system is, still in second order form,

$$\ddot{x} + B\dot{x} + Ax = 0 \quad (2.18)$$

with  $B$  defined by

$$B\dot{x} = \gamma(\dot{x}, b)b. \quad (2.19)$$

With  $y = \dot{x}$ , one may consider the equivalent first order system in  $XX$ ,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = C \begin{pmatrix} x \\ y \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}, \quad (2.20)$$

and ask: what are the eigenvalues and eigenvectors of  $C$ ? It is here that one leaves the second method of Liapounov and returns to his first. In [32] a perturbation analysis is carried out, valid for small values of  $\gamma$  in (2.17), (2.19). It is shown that, under the separation assumption

$$w_{k+1} - w_k \geq d > 0, \quad w_k = \sqrt{\lambda_k}, \quad (2.21)$$

the eigenvalues of  $C$ , which for  $\gamma = 0$  are  $\pm iw_k$ ,  $k = 1, 2, 3, \dots$ , all have negative real parts for  $\gamma > 0$  and, moreover, designating the perturbed eigenvalues by  $\zeta_k(\gamma)$ ,  $k = \pm 1, \pm 2, \pm 3, \dots$ ,  $\zeta_k(0) = iw_k$ ,  $\zeta_{-k}(0) = -iw_k$ , we have (cf. (2.21))

$$\zeta_k(\gamma) = iw_k - \frac{\gamma}{2} |b_k|^2 + o(\gamma^2 \frac{1}{|w_k|^2}), \quad k \rightarrow \infty \quad (2.22)$$

$$\zeta_{-k}(\gamma) = -i\omega_k - \frac{\gamma}{2} |b_k|^2 + o(\gamma^2 \frac{1}{|w_k|^2}), \quad k \rightarrow \infty. \quad (2.22)$$

It is also possible to show that the perturbed eigenvectors continue to form a Riesz basis for the space  $X \times X$ . From this it follows that all solutions of (2.18) tend strongly to zero in the energy norm, though not at a uniform exponential rate.

Following Wonham's initial results [48] on the finite dimensional case, there has been considerable interest displayed in the question of spectral determination via linear feedback for distributed parameter systems. In terms of the system (2.14), equivalently,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u, \quad (2.23)$$

with initial ( $u=0$ ) eigenvalues  $\pm i\omega_k$ ,  $k = 1, 2, 3, \dots$ , the question may be phrased as follows: we suppose use of a linear feedback functional

$$u = (A^{\frac{1}{2}}x, k_1) + (y, k_2), \quad k_1, k_2 \in X, \quad (2.24)$$

bounded relative to the energy norm  $(x, Ax) + (y, y) = (A^{\frac{1}{2}}x, A^{\frac{1}{2}}x) + (y, y)$  in  $X \times X$ . With

$$K_1 x = (A^{\frac{1}{2}}x, k_1)b, \quad K_2 y = (y, k_2)b \quad (2.25)$$

the closed loop system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A + K_1 & K_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.26)$$

One can now ask: What eigenvalues can be achieved for the closed loop system (2.24) by appropriate selection of  $k_1, k_2$  in (2.25)? For some time the author was under the impression that his approach via canonical forms [35] (more on this below) was the first treatment of this question but, in fact, it appears that this credit must go to Prof. Sun S.-H. of Szechuan University who treated this problem by a more sophisticated application of the perturbation technique used by the author in [32] to obtain the result. Sun was able to show, with an assumption similar to (2.21) and the Riesz basis assumption on the open loop eigenvectors, that the totality of spectra,  $\sum$ , achievable by use of (2.25) coincides with sequences  $\zeta_k, \zeta_{-k}$ ,  $k = 1, 2, 3, \dots$  for which, assuming the  $b_k \neq 0$  as before,

$$\sum_{k=1}^{\infty} \left( \left| \frac{\zeta_{2-iw_k}}{b_k} \right|^2 + \left| \frac{\zeta_{-k+iw_k}}{b_k} \right|^2 \right) < \infty.$$

His very important paper has been translated by Ho L.-F. in [44]. Some comparable, but necessarily weaker, results have been obtained by Reid in his thesis [29] for the equation of linear surface waves where (2.21) is not satisfied and, in fact,  $\lim_{k \rightarrow \infty} (w_{k+1} - w_k) = 0$ . Other results in this direction, for hyperbolic systems of various types, have been obtained by Clark [5], [6] and by Ho in his thesis [16].

Much of the initial impetus for the study of control canonical forms, both for finite and infinite dimensional systems, came from the spectral determination question discussed above, but the subject is interesting in its own right and shows some promise of being adaptable for "real world" control implementation. The reader will recall that a finite dimensional controllable system

$$\dot{x} = Ax + bu, \quad x \in \mathbb{R}^n,$$

with scalar control  $u$  is equivalent, via a state space similarity transformation (see [20], [35]) to a system in rational canonical form corresponding to the  $n$ -th order scalar equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = u, \quad (2.27)$$

where

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n$$

is the characteristic polynomial of the matrix  $A$ . Comparable, but somewhat more intricate, results are available for systems with higher control dimension [20], [2]. In [38] we note that if one employs a scalar linear observation

$$y = h^* x = (x, h), \quad (2.28)$$

there is exactly one observation vector  $h \in \mathbb{R}^n$  for which (2.28) satisfies (2.27); for general  $h$  the right hand side will involve the derivatives of  $u$  of order  $\leq u - 1$ . Systems (2.27) are particularly easy to deal with. Closed loop eigenvalues  $\zeta_1, \zeta_2, \dots, \zeta_n$  may be realized simply by forming the polynomial

$$q(\lambda) = \prod_{k=1}^n (\lambda - \zeta_k) = \lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n$$

and determining  $u$  by linear feedback on the observation  $y$  and its

derivatives,

$$u = \sum_{k=1}^n (a_k - c_k) y^{(n-k)}.$$

Apparently less well known, but quite obvious, is that the control problem for (2.27) is, in a sense, trivial. Let us suppose the initial instant is taken to be  $t = 0$  and control is to be effected during  $0 \leq t \leq T$ . Let the initial state be specified by

$$y^{(n-k)}(0) = y_{n-k+1}, \quad k = 1, 2, \dots, n \quad (2.29)$$

and the terminal state by

$$y^{(n-k)}(T) = Y_{n-k+1}, \quad k = 1, 2, \dots, n. \quad (2.30)$$

If  $y(t)$  satisfies (2.29) and

$$y^{(n)}(t) = v(t), \quad 0 \leq t \leq T \quad (2.31)$$

then we see readily that for  $k = 1, 2, \dots, n$

$$y^{(n-k)}(t) = \sum_{\ell=1}^k y_{n-\ell+1} \frac{t^{k-\ell}}{(k-\ell)!} + \int_0^t (t-s)^{k-1} v(s) ds$$

and (2.30) is achieved just in case

$$\int_0^T (t-s)^{k-1} v(s) ds = Y_{n-k+1} - \sum_{\ell=1}^k Y_{n-\ell+1} \frac{T^{k-\ell}}{(k-\ell)!},$$

$$k = 1, 2, \dots, n.$$

This is easily solved for  $v$  in various function classes, e.g. polynomials of degree  $\leq n-1$ , etc. and, it should be noted, the solution has nothing to do with the coefficients in (2.27) so the calculation can be carried out once for any given  $T$  and recorded for use ever after. Then in a given canonical system (2.27) we need only set

$$u(t) = v(t) - \sum_{k=1}^n a_k y^{(n-k)}(t) \quad (2.32)$$

to realize the desired control objective.

Since, in a given control context, it is not likely that the available observation (2.28) will be the particular one for which (2.27) obtains, the above result might seem to be a generally useless curiosity. It turns out, however, that in canonical form theory there is a counterpart to the more widely known observer theory. If  $C$  is any  $n \times n$  matrix whose minimal and

characteristic polynomials coincide. it is possible to select (non-uniquely)  $r$ ,  $d$  and  $j$  such that the augmented system

$$\dot{x} = Ax + bu \quad (2.33)$$

$$\dot{z} = ry + Cz + du \quad (= rh^*x + Cz + du \quad \text{since } y = h^*x) \quad (2.34)$$

with augmented observation

$$w = y + j^*z = h^*x + j^*z \quad (2.35)$$

is in canonical form, so that for some coefficients  $\alpha_1, \alpha_2, \dots, \alpha_{2m}$

$$w^{(2n)} + \alpha_1 w^{(2n-1)} + \dots + \alpha_{2m-1} w' + \alpha_{2m} w = u.$$

The adjointed system (2.34) can be realized electronically, just as an observer system is, and the considerable freedom in choice of  $C$ ,  $r$ ,  $d$  and  $j$  provides much design flexibility. In some cases the dimension of (2.34) can be reduced. The proof that (2.33), (2.34), (2.35) can be made a canonical system appears in [38].

A parallel control canonical form theory has been developed for certain hyperbolic distributed parameter systems, involving neutral functional equations in place of the  $n$ -th order scalar equation (2.27). The theory is quite complex, especially as it applies to partial differential equations with variable coefficients (see [35], [16], [38], [39] e.g.). To give an idea how the theory is developed we will consider the constant coefficient case of (1.4) which, without loss of generality, we can take to be

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0, \quad t \geq 0, \quad 0 \leq x \leq 1, \quad (2.36)$$

$$w(0, t) = 0, \quad \frac{\partial w}{\partial x}(1, t) = u(t). \quad (2.37)$$

The normalized eigenfunctions of the corresponding homogeneous system are

$$\phi_k(x) = \sqrt{2} \sin \frac{2k-1}{2} \pi x, \quad k = 1, 2, 3, \dots \quad (2.38)$$

Setting  $w_k = \frac{2k-1}{2} \pi$  and forming the expansions

$$w(x, t) = \sum_{k=1}^{\infty} w_k(t) \phi_k(x), \quad (2.39)$$

$$\frac{\partial w}{\partial t}(x, t) = \sum_{k=1}^{\infty} v_k(t) \phi_k(x), \quad (2.40)$$

followed by the transformation

$$\begin{pmatrix} w_k \\ v_k \end{pmatrix} = \begin{pmatrix} \frac{1}{i w_k} & -\frac{1}{i w_k} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \quad (2.41)$$

we have, for  $k = 1, 2, 3, \dots$

$$\dot{\eta}_k = i w_k \eta_k + \frac{(-1)^{k-1}}{\sqrt{2}} u(t), \quad \dot{\zeta}_k = -i w_k \zeta_k + \frac{(-1)^{k-1}}{\sqrt{2}} u(t). \quad (2.42)$$

Consider now the neutral delay equation

$$y(t+2) + y(t) = u(t+2). \quad (2.43)$$

The characteristic function of the homogeneous equation is

$$p(\lambda) = e^{2\lambda} + 1 = 2e^{\lambda} \cosh \lambda$$

and the zeros of  $p(\lambda)$  are precisely the eigenvalues  $\pm i w_k$  appearing in (2.42). The transfer function for (2.43) is

$$T_0(\lambda) = \frac{e^{2\lambda}}{e^{2\lambda} + 1} = \frac{1}{2} \frac{\sinh \lambda}{\cosh \lambda} + \frac{1}{2} \quad (2.44)$$

which can be rewritten as

$$T_0(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda}{\lambda^2 + \omega_k^2} + \frac{1}{2}, \quad \omega_k = \frac{2k-1}{2} \pi. \quad (2.44)$$

If we define an observation  $y(t)$  on (2.42) by

$$y(t) = \sum_{k=1}^{\infty} [h_k \eta_k(t) + g_k \zeta_k(t)] + \frac{1}{2} u(t)$$

the transfer function for  $y$  is, formally,

$$\sum_{k=1}^{\infty} \left[ \frac{h_k (-1)^{k-1} \frac{1}{\sqrt{2}}}{\lambda - i \omega_k} + \frac{g_k (-1)^{k-1} \frac{1}{\sqrt{2}}}{\lambda + i \omega_k} \right] + \frac{1}{2}$$

which may be seen to agree with (2.44) just in case

$$h_k = g_k = \frac{(-1)^k}{\sqrt{2}}.$$

Using (2.38), (2.39), (2.41), (2.42) it may be seen that this choice of  $h_k, g_k$  corresponds to

$$y(t) = \frac{1}{2} \frac{\partial w}{\partial t}(1, t) + \frac{1}{2} u(t) = \frac{1}{2} \left( \frac{\partial w}{\partial t}(1, t) + \frac{\partial w}{\partial x}(1, t) \right). \quad (2.45)$$

This observation on (2.42), and no other, satisfies the scalar equation (2.43) which serves as the control canonical form for (2.42). The details of the above



calculations and some idea of the form of a general theory appear in [38] and [39].

If the canonical observation (2.45) were actually available, so that we have (2.43), its usefulness is quite clear. For, with the causal feedback law

$$u(t+2) = (1-\gamma)y(t) - \int_0^2 c(s)y(t+s)ds \quad (2.46)$$

(2.43) transforms to

$$y(t+2) + \gamma y(t) + \int_0^2 c(s)y(t+s)ds = 0 \quad (2.47)$$

and it is known from [35], [44] that the exponential solution  $e^{\zeta_k t}$ ,  $e^{\zeta_{-k} t}$  of (2.47) can be made such that

$$\zeta_k = i\omega_k + \alpha + \varepsilon_k, \quad \zeta_{-k} = i\omega_k + \alpha + \varepsilon_{-k},$$

where  $\alpha$  is a complex number (ordinarily negative) determined by  $\gamma$  and  $\varepsilon_k$ ,  $\varepsilon_{-k}$  are arbitrary complex numbers, determined by  $c \in L^2[0,2]$ , such that

$$\sum_{k=1}^{\infty} (|\varepsilon_k|^2 + |\varepsilon_{-k}|^2) < \infty.$$

It may be shown that these are the eigenvalues of the closed loop system (2.36), (2.37), (2.45), (2.46).

In a given application, however, it is entirely likely that the particular "canonical" observation (2.46) will not be available. Indeed, in the example indicated, since this observation is taken at the same point where control is applied and might, therefore, be subject to a certain amount of noise disturbance, it might not be desirable to use this observation in practice. To illustrate the use of the technique of canonical augmentation (or "canonical compensation", perhaps) let us consider the same system (2.36), (2.37), but suppose the available observation is

$$y(t) = \frac{\partial w}{\partial x}(0, t). \quad (2.48)$$

It is not hard to show in this case that  $y(t)$  satisfies

$$y(t+2) + y(t) = u(t+1) \quad (2.49)$$

rather than (2.43). This "central" control canonical form is not as usable as the "backward" form (2.43) because, unlike (2.46),

$$u(t+1) = (1-\gamma)y(t) - \int_0^2 c(s)y(t+s)ds$$

is not a causal feedback law and cannot be implemented. But now couple (2.49) with

$$z(t+2) + \rho z(t) = au(t+2) + bu(t+1) + cy(t+1) + dy(t) \quad (2.50)$$

and let

$$w(t) = y(t) + z(t).$$

One ordinarily will take  $|\rho| < 1$  so that the homogeneous part of (2.50) is asymptotically stable, thus avoiding the growth of parasitic solutions in the compensator. Since

$$[y(t+4) + y(t+2) - u(t+3)] + \rho[y(t+2) + y(t) - u(t+1)] = 0$$

while

$$[z(t+4) + \rho z(t+2) - au(t+4) - bu(t+3) - cu(t+1) - cy(t+3) - dy(t+2)]$$

$$+ [z(t+2) + \rho z(t) - au(t+2) - bu(t+1) - cy(t+1) - dy(t)] = 0$$

we find that

$$w(t+4) + (1+\rho)w(t+2) + \rho w(t) = au(t+4) + [1+b]u(t+3)$$

$$+ au(t+2) + [\rho+b]u(t+1) + c[y(t+3) + y(t+1)]$$

$$+ d[y(t+2) + y(t)] = (\text{using (2.49)})$$

$$au(t+4) + [1+b]u(t+3) + [a+c]u(t+2) + [\rho+b+d]u(t+1).$$

Then it is easy to see that with

$$a = 1, \quad 1+b = a+c = \rho+b+d = 0,$$

i.e. with

$$a = 1, \quad b = -1, \quad c = -1, \quad d = 1 - \rho,$$

we arrive at the "backward" canonical form satisfied by  $w(t)$ :

$$w(t+4) + (1+\rho)w(t+2) + \rho w(t) = u(t+4)$$

for which causal feedback laws

$$u(t+4) = -\gamma_1 w(t+3) + [1+\rho - \gamma_2] w(t+2) - \gamma_3 w(t+1) + [\rho - \gamma_4] w(t) - \int_0^4 c(s) w(t+s) ds \quad (2.51)$$

may be implemented, yielding overall closed loop systems

$$w(t+4) + \gamma_1 w(t+3) + \gamma_2 w(t+2) + \gamma_3 w(t+1) + \gamma_4 w(t) + \int_0^4 c(s) w(t+s) ds = 0. \quad (2.52)$$

It is necessary to check separately that the system (2.49), (2.50), (2.51) is observable in any given case.

The exponential solutions of (2.52), and hence the eigenvalues of (2.36), (2.37), (2.48), (2.50), (2.51) may be determined with the same flexibility as already noted for (2.47). This is discussed in some detail in the thesis of R. G. Teglás [45]. A complete theory of canonical compensation for hyperbolic systems remains to be developed but, we hope, the example given here gives reason to believe that the method is a promising one. It is clear that there are some connections with observer theory as developed in [22] and elsewhere; these connections remain to be worked out.

#### REFERENCES

1. Birkhoff, G. D., and G. -C. Rota Ordinary Differential Equations. Ginn and Co., New York, 1959.
2. Brunovsky, P., "A Classification of Linear Controllable Systems," *Kybernetika*, Vol. 6, 1970, pp. 173-188.
3. Butkovskiy, A. G., "The Method of Moments in the Theory of Optimal Control of Systems with Distributed Parameters," *Automat. Rem. Contr.*, Vol. 24, 1963, pp. 1106-1113.
4. \_\_\_\_\_, *Theory of Optimal Control of Distributed Parameter Systems*. American Elsevier Pub. Co., New York, 1969.
5. Clarke, B. M. N., "Control Canonical Forms and Eigenvalue Assignment by Feedback for a Class of Linear Hyperbolic Systems," *SIAM J. Cont. and Opt.*, Vol. 19, 1981, pp. 711-729.
6. \_\_\_\_\_, *Eigenvalue Assignment of an Augmented Hyperbolic System by Linear Feedback*, Macquarie Mathematics Reports, No. 80-0017, School of Mathematics and Physics, Macquarie University, North Ryde, Australia, 1980.
7. Coddington, E. H., and N. Levinson, *Theory of Ordinary Differential Equations*. McGraw Hill Book Co., New York, 1955.
8. Courant, R., and D. Hilbert, *Methods of Mathematical Physics, Vol II: Partial Differential Equations*. Interscience Pub. Co., New York, 1962.
9. Dafermos, C. M., "Wave Equations with Weak Damping," *SIAM J. Appl. Math.*, Vol. 18, 1970, pp. 759-767.

10. Datko, R., "A Linear Control Problem in an Abstract Hilbert Space," *J. Diff. Eq'ns*, Vol. 9, 1971, pp. 346-359.
11. Egorov, Yu. V., "Some Problems in the Theory of Optimal Control," *Soviet Mathematics*, Vol. 3, 1962, pp. 1080-1084.
12. \_\_\_\_\_, "Optimal Control on Banach Space," *Ibid.*, Vol. 4, 1963.
13. Fattorini, H. O., "Control in Finite Time of Differential Equations in Banach Space," *Comm. Pure Appl. Math.*, Vol. 19, 1966, pp. 17-34.
14. Gal'chuk, L. I., "Optimal Control of Systems Described by Parabolic Equations," *SIAM J. Control*, Vol. 7, 1969, pp. 546-558.
15. Herget, C. J., "On the Controllability of Distributed Parameter Systems," *Inter. J. Control*, Vol. 11, 1970, pp. 827-833.
16. Ho, L. F., "Controllability and Spectral Assignability of a Class of Hyperbolic Control Systems with Retarded Control Canonical Forms, Thesis, University of Wisconsin, Madison. August 1981.
17. Ho, L.F., and D. L. Russell, "Admissible Input Elements for Linear Systems in Hilbert Space and a Carleson Measure Criterion. To appear.
18. Ingham, A. E., "Some Trigonometrical Inequalities in the Theory of Series," *Math. Zeitschr.*, Vol. 41, 1936, pp. 367-379.
19. LaSalle, J. P., and S. Lefschetz, *Stability by Liapounov's Direct Method with Applications*. Academic Press, New York, 1961.
20. Lee, E. B., and L. W. Markus, *Foundations of Optimal Control Theory*. Wiley, New York, 1967.
21. Levinson, N., "Gap and Density Theorems," *Amer. Math. Soc. Colloquium Pub.*, Vol. 26, 1940.
22. Luenberger, D. G., "An Introduction to Observers," *IEEE Trans. Auto. Control*, Vol. AC-16, 1971, pp. 596-602.
23. Mizel, V. J., and T. I. Seidman, "Observation and Prediction for the Heat Equation," *J. Math. Anal. Appl.*, Vol. 28, 1969, pp. 303-312. Part II: *Ibid.*, Vol. 38, 1972, pp. 149-166.
24. Paley, R. E. A. C., and N. Wiener, "Fourier Transforms in the Complex Domain," *Amer. Math. Soc. Coll. Pub.*, Vol. 19, Providence, 1934.
25. Petrovski, I. G., *Partial Differential Equations*. Iliffe Books, London, 1967.
26. Pontryagin, L. S., V. G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The Mathematical Theory of Optimal Processes*. Interscience Pub., New York, 1962.

27. Quinn, J. P., Time Optimal Control of Linear Distributed Parameter Systems, Thesis, University of Wisconsin, Madison, August 1969.
28. Quinn, J. P., and D. L. Russell, "Asymptotic Stability and Energy Decay Rates for Solutions of Hyperbolic Equations with Boundary Damping," Proc. Royal Soc. Edinburgh, Vol. 77A, 1977, pp. 7-12.
29. Reid, R. M., Some Controllability and Stabilizability Properties of Linear Water Waves, Thesis, University of Wisconsin, Madison, August 1979.
30. Russell, D. L., On Boundary Value Controllability of Linear Symmetric Hyperbolic Systems, in Mathematical Theory of Control, Academic Press, New York, 1967, pp. 312-321.
31. \_\_\_\_\_, "Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems," J. Math. Anal. Appl., Vol. 18, 1967, pp. 542-559.
32. \_\_\_\_\_, "Linear Stabilization of the Linear Oscillator in Hilbert Space," J. Math. Anal. Appl., Vol. 25, 1969, pp. 663-675.
33. \_\_\_\_\_, "Decay Rates for Weakly Damped Systems in Hilbert Space Obtained With Control-Theoretic Methods," J. Diff. Eqns., Vol. 19, 1975, pp. 344-370.
34. \_\_\_\_\_, "Control Theory of Hyperbolic Equations Related to Certain Questions in Harmonic Analysis and Spectral Theory," J. Math. Anal. Appl., Vol. 40, 1972, pp. 336-368.
35. \_\_\_\_\_, "Canonical Forms and Spectral Determination for a Class of Hyperbolic Distributed Parameter Control Systems," Ibid., Vol. 62, 1978, pp. 186-225.
36. \_\_\_\_\_, "Controllability and Stabilization Theory for Linear Partial Differential Equations: Recent Progress and Open Questions," SIAM Review, Vol. 20, 1978, pp. 639-739.
37. \_\_\_\_\_, Closed Loop Eigenvalue Specification for Augmented and Deficient Hyperbolic Systems, Technical Summary Report 2021. Mathematics Research Center, University of Wisconsin, Madison, 1979.
38. \_\_\_\_\_, Control Canonical Structure for a Class of Distributed Parameter Systems, Proceedings of Third IMA Conf. on Control Theory, Sheffield, 1980.
39. \_\_\_\_\_, Functional Equations as Control Canonical Forms for Distributed Parameter Systems and a State Space Theory for Certain Differential Equations of Infinite Order, Proceedings Conf. on Functional Eq'ns. and Control Theory, Blacksburg, Va., 1981, to appear.
40. Sakawa, Y., "Observability and Related Problems for Partial Differential Equations of Parabolic Type," SIAM J. Control, Vol. 13, 1975, pp. 14-27.

41. \_\_\_\_\_, Feedback Control of Second Order Evolution Equations with Damping, to appear.
42. Schwartz, L., Etude des sommes d'exponentielles. deuxième édition, Hermann, Paris, 1959.
43. Slemrod, M., "Asymptotic Behavior of a Class of Abstract Dynamical Systems," J. Diff. Eq'ns., Vol. 7, 1970, pp. 584-600.
44. Sun, S. -H., "On Spectrum Distribution of Completely Controllable Linear Systems," SIAM J. Contr. and Opt., Vol. 19, 1981, pp. 730-743.
45. Teglasi, R. G., A Control Canonical Form for a Class of Linear Hyperbolic Systems. Thesis, University of Wisconsin, Madison, June 1981.
46. Triggiani, R., "On the Stabilizability Problem in Banach Space," J. Math. Anal. Appl., Vol. 52, 1975, pp. 383-403.
47. Weck, N., "A Remark on Controllability for Symmetric Hyperbolic Systems in one Space Dimension," SIAM J. Contr. and Opt., Vol. 20, 1982, pp. 1-8.
48. Wonham, W. M., "On Pole Assignment in Multi-input Controllable Linear Systems," IEEE Trans. Auto. Contr., Vol. AC-12, 1967, pp. 660-665.
49. Young, R. M., An Introduction to Non-harmonic Fourier Series. Academic Press, New York, 1980.

Admissible Input Elements  
for Systems in Hilbert Space  
and a Carleson Measure Criterion<sup>\*</sup>

by L.F. Ho<sup>+</sup> and D. L. Russell<sup>++</sup>

Abstract

We study the control system

$$\dot{x} = Ax + bu, \quad x \in X, \quad u \text{ scalar,}$$

where  $A$  generates a semigroup on the Hilbert space  $X$  but, in general, the control input element  $b \notin X$ . Many boundary value control systems, point control force situations, etc., can be studied in this context. We define and analyze "admissible" input elements  $b$  and develop sufficient conditions for  $b$  to be admissible in terms of the Carleson measure theorem of  $H^p$ -theory.

<sup>\*</sup>Supported in part by the Air Force Office of Scientific Research under Grant AFOSR 79-0018.

<sup>+</sup>Department of Mathematics, University of Oklahoma, Norman, Oklahoma.

<sup>++</sup>Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706.

1. Introduction. One commonly studies linear, time invariant control systems in a Banach space  $X$  in the form

$$\dot{x} = Ax + Bu, \quad x \in X, \quad u \in U, \quad (1.1)$$

where  $A$  is the generator of a strongly continuous semigroup of bounded operators  $\{S(t) | t \geq 0\}$  on  $X$  and  $B$  is a bounded operator from the control space,  $U$ , into  $X$ . If  $u: [0, \infty) \rightarrow U$  is locally (Bochner) integrable, generalized (or "mild") solutions of (1.1) corresponding to an initial state

$$x(0) = x_0 \in X$$

can be represented by the "variation of parameters" formula (see, e.g. [3], [11])

$$\dot{x}(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds \quad (1.2)$$

and a number of properties of  $x(t)$  thereby deduced.

It is well known, however, that most of the "interesting" infinite dimensional control systems do not arise this way because the degree of controllability of a system (1.1) with  $B$  bounded is rather restricted if, as is usually the case,  $U$  is finite dimensional or for some other reason the operator  $B$  is compact. Indeed, most of the mathematically intriguing examples arise in the context of partial differential equations with boundary value control inputs, control forces exerted at isolated points, etc., and in the context of functional equations which involve values of the control of discrete instants, viz.;  $u(t), u(t-T_1), \dots, u(t-T_n)$ . In each of these cases the formulation (1.1) is inadequate and one must consider input



operators  $B$  whose range is not restricted to the space  $X$ .

A number of authors have addressed the problem of interpretation of (1.1) for operators  $B$  of rather general type. We particularly cite the contributions of Curtain and Pritchard [3], Zabczyk [22], Fattorini [6], and Washburn [20]. It seems fair to say that, as brought out in [3], the theory is more extensive and generally applicable in the case of systems of "diffusion type", ordinarily involving holomorphic semigroups, than in systems of "wave" or hyperbolic character.

In the present article we shall restrict our attention to spaces  $X$  which are separable Hilbert spaces and to finite dimensional control spaces  $U$ .

Taking  $U$  to be  $R^m$ , (1.1) becomes

$$\dot{x} = Ax + \sum_{j=1}^m b_j u^j \quad (1.2)$$

where  $b_j$  is the control input element associated with the  $j$ -th control component  $u^j$ . Since every solution of (1.2) is a linear combination of solutions of  $\dot{x} = Ax$  and the individual systems  $\dot{x} = Ax + b_j u^j$ ,  $j = 1, 2, \dots, m$ , we may, without loss of generality, confine our discussion to systems

$$\dot{x} = Ax + bu \quad (1.3)$$

wherein the control  $u$  is scalar valued. Much of our theory can be extended to cases wherein  $U$  is infinite dimensional but we will not do that here.

What distinguishes the present study from earlier contributions is the attention which we pay not only to the relationship between the operator,  $A$ , and the input element  $b$ , but also to the relationship between  $b$  and the semigroup  $S(t)$  generated by  $A$ . In cases where  $A$  has discrete

spectrum  $\{\lambda_k | k \in K\}$ ,  $K$  being a countable index set, this amounts to a study encompassing the input element  $b$ , the eigenvectors  $\{\phi_k | k \in K\}$  of  $A$ , the corresponding eigenvectors of the dual operator,  $A'$  as defined in Section 2, and the exponential functions  $\exp(\lambda_k t)$ ,  $k \in K$ . It is in particular reference to the latter that what is probably the most important idea of this paper is developed. We show that a sufficient condition for  $b$  to be an "admissible input element" (definition in Section 2) can be given in terms of a measure on Borel subsets of the complex plane whose support is  $\{-\lambda_k | k \in K\}$ . When that measure turns out to be a Carleson measure the input element  $b$  is admissible. This result brings out yet again the intimate relationship between the control theory of infinite dimensional linear systems and parallel developments in  $H^p$  theory ([5], [8], [12]) and the related theory of completeness and independence of sets of complex exponentials.

## 2. Admissible Input Elements.

Let  $X$  be a separable Hilbert space and let  $A$  be a closed operator on  $X$  with domain,  $\mathfrak{D}(A)$ , dense in  $X$ , generating a strongly continuous semigroup of bounded operators  $S(t)$  on  $X$  for  $t \geq 0$ . For  $b \in X$  the (generalized, or "mild") solution of

$$\dot{x} = Ax + bu, \quad u \in L_{loc}^2[0, \infty), \quad (2.1)$$

$$x(0) = x_0 \in X, \quad (2.2)$$

is given by the "variation of parameters" formula

$$x(t) = S(t)x_0 + \int_0^t S(t-s)bu(s)ds \quad (2.3)$$

and may be seen to be a continuous function  $x : [0, \infty) \rightarrow X$ . Whether  $\dot{x}(t)$  is defined for each  $t \geq 0$  and (2.1) holds is more complicated: sufficient conditions are that  $b \in \mathfrak{D}(A)$  or that  $u$  is differentiable as a function of  $t$  ([3], [11]).

In this paper we wish to consider (2.1), (2.2) in certain cases where  $b$  does not lie in  $X$  and to provide, for such  $b$ , a formula parallel to (2.3). Our approach is similar to that used in [14].

Identifying  $X$  with its dual  $X'$ , we denote the duality relationship by  $\langle x, y \rangle$ ,  $x \in X$ ,  $y \in X$ , linear in both  $x$  and  $y$ . Where  $X$  is the complexification of a real Hilbert space  $X_0$  the conjugate element  $\bar{y}$  is well defined for each  $y \in X$  and, with  $(\cdot, \cdot)$  denoting the inner product in  $X$ ,

$$\langle x, y \rangle = \langle x, \bar{y} \rangle, \quad \langle x, y \rangle = (x, \bar{y}).$$

The bilinear form  $\langle \cdot, \cdot \rangle$  is symmetric, i.e.,  $\langle x, y \rangle = \langle y, x \rangle$ ,  $x, y \in X$ ,

and, for all  $x \in X$ ,

$$\|x\|_X = \sup_{\substack{y \in X \\ y \neq 0}} \frac{|\langle x, y \rangle|}{\|y\|_X}. \quad (2.4)$$

The symbol  $A'$  will be used to denote the dual of  $A$  relative to the bilinear form  $\langle \cdot, \cdot \rangle$ , that is

$$\langle Ax, y \rangle = \langle x, A'y \rangle, \quad x \in \mathcal{D}(A), \quad y \in \mathcal{D}(A').$$

The operator  $A'$  is closed with domain  $\mathcal{D}(A')$  dense in  $X$ . It is known that if  $A$  generates a semigroup  $S(t)$ , then  $S(t)'$  is also a semigroup, generated by  $A'$ . See [4] for details.

Let  $Y$  be a dense subspace of  $X$  which is a Hilbert space in its own right with norm  $\|\cdot\|_Y$  stronger than  $\|\cdot\|_X$  so that the injection map

$$j = Y \rightarrow X \\ j(y) = y, \quad y \in Y,$$

is one-to-one and continuous with dense range  $Y \subset X$ . We further suppose that  $Y$  is invariant under the action of  $S(t)'$ :  $y \in Y \Rightarrow S(t)'y \in Y$ , and that this map is continuous with respect to  $\|S(t)'y\|_Y$ ,  $\|y\|_Y$  and the usual topology of  $[0, \infty)$ .

Let  $Y'$  be the dual of  $Y$  with respect to  $X$  as described, e.g. in [1], [14], [15]. This means that  $Y'$  is the closure of  $X$  with respect to the norm

$$\|x\|_{Y'} = \sup_{\substack{y \in Y \\ y \neq 0}} \frac{|\langle x, y \rangle|}{\|y\|_Y} \quad (2.5)$$

It is known that  $Y'$ , so defined, is a realization of the dual space of  $Y$  and it is easily verified that the bilinear form  $\langle x, y \rangle$  may be defined, by continuity, for  $x \in Y'$ ,  $y \in Y$  as

$$\langle x, y \rangle = \lim_{k \rightarrow \infty} \langle x_k, y \rangle$$

where  $\{x_k\}$  is a sequence in  $X$  converging to  $x$  in  $\|\cdot\|_{Y'}$ . So defined,  $\langle x, y \rangle$  generates, as  $x$  ranges over  $Y'$ , all continuous linear functionals on  $Y$ . We have

$$X \subset X \subset Y'.$$

Definition 2.1. In the system (2.1), i.e.,

$$\dot{x} = Ax + bu, \quad u \in L^2_{\text{loc}}[0, \infty),$$

$b$  is an admissible input element if there exist  $Y, Y'$ , as above, with  $b \in Y'$ , such that for every  $T > 0$  the continuous map

$$L_T : Y \rightarrow C[0, T]$$

defined by

$$(Ly)(t) = \langle b, S(t)'y \rangle, \quad y \in Y, \quad t \in [0, T], \quad (2.6)$$

has a continuous extension to

$$L_T : X \rightarrow L^2[0, T].$$

Remark. It is clear that this amounts to the statement that in the dual observed system

$$\begin{aligned} \dot{y} &= A'y \\ z &= \langle b, y \rangle, \end{aligned}$$

$b$  is an admissible observation element; that is, for  $y \in Y$ ,

$$z(\cdot) = \langle b, S(\cdot)'y \rangle \in C[0, T],$$

this relationship extending continuously to  $z(\cdot) \in L^2[0, T]$  for  $y \in X$ .

To verify that Definition 2.1 enables consistent definition, at least in a generalized sense, of solutions of (2.1), (2.2) when  $b$  is an admissible input element and to establish some of the properties of the resulting solution, we present

Theorem 2.2. If  $b$  is an admissible input element, the formula

$$\langle x(t), y \rangle = \langle x_0, S(t)'y \rangle + \int_0^t \langle b, S(t-s)'y \rangle u(s) ds, \quad y \in Y, \quad (2.7)$$

defines, for each  $t \geq 0$ , a unique element  $x(t) \in X$ . Given  $T > 0$  and  $u \in L^2[0, T]$

$$x(t) = S(t)x_0 + B(t)u, \quad t \in [0, T], \quad (2.8)$$

where  $B(t)$  is the strongly continuous family of bounded operators  $B(t) : L^2[0, T] \rightarrow X$  given by

$$\langle B(t)u, y \rangle = \int_0^t \langle b, S(t-s)'y \rangle u(s) ds, \quad y \in Y. \quad (2.9)$$

Proof. From (2.8) and the fact that  $Y$  is dense in  $X$  it is clear that

$$x(t) - S(t)x_0 \equiv \xi(t) \equiv B(t)u$$

where, for  $y \in Y$ ,

$$\langle \xi(t), y \rangle = \int_0^t \langle b, S(t-s)'y \rangle u(s) ds.$$

Let  $x \in X$  and let  $\{y_k\}$  be a sequence in  $Y$  converging to  $x$  with respect to  $\|\cdot\|_X$ . Since  $b$  is an admissible input element the

corresponding functions  $h_k$  defined by

$$h_k(t-s) = \langle b, S(t-s)' y_k \rangle \quad (2.10)$$

converge in  $L^2[0, T]$  to a function  $h \in L^2[0, T]$ . Defining

$$\langle \xi(t), x \rangle = \int_0^t h(t-s) u(s) ds,$$

we see that for  $t \in [0, T]$

$$\begin{aligned} |\langle \xi(t), x \rangle| &\leq \|h\|_{L^2[0, T]} \|u\|_{L^2[0, T]} \\ &\leq \|L_T\| \|x\|_X \|u\|_{L^2[0, T]} \end{aligned}$$

since (cf. (2.6), (2.10))  $h = L_T x$ . Hence  $\xi(t) \in X' = X$ . This also gives

$$\|\xi(t)\|_X \leq \|L_T\| \|u\|_{L^2[0, T]},$$

showing that for  $t \in [0, T]$ ,  $B(t)$  is bounded with

$$\|B(t)\| \leq \|L_T\|.$$

To establish that  $\xi(t)$  is continuous in  $t$  for each fixed  $u \in L^2[0, T]$  (and, hence, that  $B(t)$  is strongly continuous in  $t$ ), let  $0 \leq t \leq \hat{t} \leq T$  and form, for  $y \in Y$

$$\begin{aligned} \langle \xi(\hat{t}) - \xi(t), y \rangle &= \int_0^{\hat{t}} \langle b, S(t-s)' y \rangle u(s) ds \\ &\quad - \int_0^t \langle b, S(t-s)' y \rangle u(s) ds = \text{(with } \tau = s - (\hat{t} - t)) \\ &= \int_0^{\hat{t}-t} \langle b, S(t-\tau)' y \rangle u(\tau + (\hat{t} - t)) d\tau - \int_0^t \langle b, S(t-s)' y \rangle u(s) ds \\ &\quad + \int_0^{\hat{t}-t} \langle b, S(t-s)' y \rangle u(s) ds = \end{aligned}$$

$$\begin{aligned}
&= \int_0^t \langle b, S(t-s)'y \rangle (u(s+(t-t)) - u(s)) ds + \int_0^{\hat{t}-t} \langle b, S(t-s)'y \rangle u(s) ds \\
&\leq \|L_T\| \|y\|_X (\|u(\cdot + (\hat{t}-t)) - u\|_{L^2[0,t]} + \|u\|_{L^2[0,\hat{t}-t]}).
\end{aligned}$$

Since  $Y$  is dense in  $X$  and since for fixed  $u \in L^2[0,T]$  we have

$$\lim_{\hat{t}-t \rightarrow 0} \|u\|_{L^2[0,\hat{t}-t]} = 0,$$

$$\lim_{t \uparrow \hat{t}} \|u(\cdot + (t-t)) - u\|_{L^2[0,t]} = \lim_{\hat{t} \downarrow t} \|u(\cdot + (\hat{t}-t)) - u\|_{L^2[0,t]} = 0.$$

We conclude that for fixed  $u \in L^2[0,T]$ , and  $t, \hat{t}$  as described,

$$\lim_{t \uparrow \hat{t}} \|\xi(\hat{t}) - \xi(t)\|_X = \lim_{\hat{t} \downarrow t} \|\xi(\hat{t}) - \xi(t)\|_X = 0$$

and thus  $\xi(t)$  is continuous in  $X$ . This completes the proof of the theorem.

Let  $H$  be a separable Hilbert space and let  $\{p_k | k \in K\}$  be a sequence in  $H$ ,  $K$  being a countable ordered index set. The  $p_k$  are strongly independent if no  $p_k$  lies in the closed span of  $\{p_l | l \neq k\}$ . If, in addition, there is a positive number  $c$  such that whenever

$$p = \sum_{K_0} \alpha_k p_k, \quad (2.11)$$

the  $\alpha_k$  being complex and  $K_0$  an arbitrary finite subset of  $K$ , we have

$$\sum_{K_0} |\alpha_k|^2 \leq c^2 \|p\|_H^2 \quad (2.12)$$

we say that the  $p_k$  are uniformly  $\ell^2$ -independent, since (2.12) implies

$$\sum_K |\alpha_k|^2 \leq c^2 \|p\|_H^2 \quad (2.13)$$



whenever  $\{\alpha_k\} \in \ell^2$  and  $p = \sum_K \alpha_k p_k$  is convergent in  $H$ .

If there is a positive number  $C$  such that

$$\|p\|_H^2 \leq C^2 \sum_{K_0} |\alpha_k|^2$$

$p$  as in (2.11), we say that the sequence  $\{p_k\}$  is uniformly  $\ell^2$ -convergent since this property implies that if  $\{\alpha_k\} \in \ell^2$  the series  $\sum_K \alpha_k p_k$  is convergent in  $H$  and

$$\|p\|_H^2 \leq C^2 \sum_K |\alpha_k|^2. \quad (2.14)$$

Recall that a sequence  $\{p_k\}$  in  $H$  forms a Schauder basis for  $H$  if for every  $p \in H$  there are unique coefficients  $\alpha_k$  such that the series  $\sum \alpha_k p_k$  converges to  $p$  in  $H$  ([21]). A Schauder basis which is, at the same time, both uniformly  $\ell^2$ -independent and uniformly  $\ell^2$ -convergent is a Riesz basis. For evident reasons we shall also use, synonymously, the term uniform  $\ell^2$ -basis. If  $\{p_k\}$  is a uniform  $\ell^2$ -basis for  $H$  then every  $p$  in  $H$  has a unique convergent representation

$$p = \sum_K \alpha_k p_k$$

with (cf. (2.13), (2.14))

$$c^{-2} \sum_K |\alpha_k|^2 \leq \|p\|_H^2 \leq C^2 \sum_K |\alpha_k|^2.$$

For the remainder of this section we suppose that

- (1) the operator  $A$  with dense domain  $\mathcal{D}(A) \subset X$  generates the strongly continuous semigroup of bounded operators  $S(t)$ ,  $t \geq 0$ ;

- (ii)  $\sigma(A)$ , the spectrum of  $A$ , consists of discrete, simple eigenvalues  $\lambda_k$ ,  $k \in K$ , and the corresponding normalized eigenvectors  $\phi_k$ ,  $k \in K$ , form a strongly independent, uniformly  $l^2$ -convergent Schauder basis for  $X$ .

Since the  $\phi_k$ ,  $k \in K$ , are strongly independent and have closed span equal to  $X$  there exist unique biorthogonal elements  $\psi_k$ ,  $k \in K$ , such that

$$\langle \psi_k, \phi_l \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}, \quad k, l \in K.$$

As is well known, the  $\psi_k$  are eigenvectors of the dual operator  $A'$  corresponding to the eigenvalues  $\lambda_k$ ,  $k \in K$ . We further assume

- (iii) the eigenvectors  $\psi_k$  of  $A'$  have the property

$$\psi_k \in Y \subset X.$$

(this is true, for example, if  $Y \supset \mathfrak{D}((A')^r)$  for some positive integer  $r$ ).

If  $x \in X$ , the fact that the  $\phi_k$  form a Schauder basis in  $X$  implies the existence of unique  $\xi_k$ ,  $k \in K$ , such that

$$x = \sum_K \xi_k \phi_k, \quad (2.15)$$

the series converging in  $X$ . From this it is evident that

$$\xi_k = \langle \psi_k, x \rangle, \quad k \in K.$$

We are not assured, in general, that the  $\xi_k$  are square summable but the uniform  $l^2$ -convergence property of the  $\phi_k$  shows the square summability of the sequence  $\{\xi_k\}$  to be a sufficient condition for convergence of (2.15).

Since we assume the  $\psi_k$  lie in  $Y$ , given any element  $b \in Y'$  (and this includes  $b \in X$ ) we may define

$$b_k = \langle \psi_k, b \rangle \quad (2.16)$$

and obtain a set of coefficients  $b_k$ ,  $k \in K$ , associated with  $b$ . In general it is not possible to recover  $b$  from the coefficients  $b_k$ . (An example is  $X = L^2[0, 2\pi]$ ,  $Y = H^1[0, 2\pi]$ ,  $\psi_k(x) = (2\pi)^{-1/2} e^{ikx}$ ,  $k = 0, 1, 2, \dots$ . The  $\psi_k (= \phi_k)$  here form an orthonormal basis for  $X$  and belong to  $Y$  but there is a non-zero element, namely  $\delta_{(0)} - \delta_{(2\pi)}$ , in  $Y'$  for which all of the  $b_k$  are zero. This arises, of course, because the closed span of the  $\psi_k$  in  $Y$  is not equal to  $Y$ .) As a consequence it is not generally meaningful to write  $b = \sum_K b_k \phi_k$ .

Nevertheless it may be meaningful to consider the initial value problem (2.1), (2.2), i.e.,

$$\begin{aligned} \dot{x} &= Ax + bu \\ x(0) &= x_0 \in X, \quad u \in L^2_{loc}[0, \infty), \end{aligned}$$

for certain  $b \in Y'$ , namely, those that we have already characterized as admissible input elements. We wish now to show that the class of such admissible input elements can be characterized in terms of the coefficients  $b_k$  and the eigenvalues  $\lambda_k$ . If  $x(t)$  is the solution of (2.1), (2.2) established by Theorem 2.2 for an admissible input element  $b$  then, in particular, for  $t \geq 0$ ,

$$\begin{aligned} \langle x(t), \psi_k \rangle &= \langle x_0, S(t)' \psi_k \rangle \\ &\quad + \int_0^t \langle b, S(t-s)' \psi_k \rangle u(s) ds \\ &= e^{\lambda_k t} x_{0,k} + \int_0^t e^{\lambda_k(t-s)} u(s) ds \end{aligned}$$

where

$$x_0 = \sum_K x_{0,k} \phi_k.$$

We do not know that the numbers  $\frac{\lambda_k^t}{e^{\lambda_k t}} x_{0,k}$  are square summable but the series

$$\sum_K e^{\lambda_k t} x_{0,k} \phi_k$$

must converge to  $S(t)x_0$  by virtue of the (assumed) Schauder basis property of the  $\phi_k$ . It follows that a sufficient condition for  $x(t)$  to belong to  $X$  is that the numbers

$$\zeta_k(t) = b_k \int_0^t e^{\lambda_k(t-s)} u(s) ds \quad (2.17)$$

should be square summable for each  $t \geq 0$ . Equivalently, making a trivial change of independent variable,

$$\zeta_k(t) = b_k \int_0^t e^{\lambda_k s} f(s) ds, \quad f(s) = u(t-s).$$

The necessity of considering an infinite number of values of  $t$  can be obviated by taking  $f$  to be an element of  $L^2[0, T]$ ,  $T > 0$  fixed, and defining  $f(s) \equiv 0$  in  $[t, T]$  for  $t < T$ . The map

$$\zeta_k = b_k \int_0^T e^{\lambda_k s} f(s) ds, \quad f \in L^2[0, T], \quad (2.18)$$

so defined may be designated as

$$L'_T : L^2[0, T] \rightarrow X, \quad (2.19)$$

$$L_T(f) = x = \sum_K \zeta_k \phi_k, \quad (2.20)$$

and it is easy to see that  $L'_T$  is the dual of  $L_T : X \rightarrow L^2[0, T]$  as defined by (2.6). Thus the boundedness of  $L_T$ , as required in

Definition 2.1, may be obtained as an immediate corollary if it is shown that  $L_T'$ , defined by (2.18) - (2.19), is bounded. For our present purpose this is the route of choice.

Extending  $f$  further via  $f(t) = 0$ ,  $t > T$ , the Laplace transform of  $f$  is the entire function

$$\phi(z) = \int_0^{\infty} e^{-zt} f(t) dt = \int_0^T e^{-zt} f(t) dt.$$

In terms of  $\phi$  we clearly have

$$\zeta_k = b_k \phi(-\lambda_k), \quad k \in K,$$

and the following proposition is evident.

Proposition 2.3.     The operator  $L_T$  (equivalently  $L_T'$ ) is bounded just in case, for every  $f \in L^2[0, T]$  the Laplace transform of  $f$ ,  $\psi$  has the property

$$\sum_K |b_k \phi(-\lambda_k)|^2 < \infty. \quad (2.21)$$

We are fortunate that the inequality can often be established with the use of the concept of a Carleson measure and the corresponding Carleson measure theorem as it applies to the space

$$H_{\alpha}^2 \equiv H^2\{z \mid \operatorname{Re}(z) > \alpha\}, \quad \alpha \text{ real}. \quad (2.22)$$

The space  $H^2\{z \mid \operatorname{Re}(z) > \alpha\}$  consists of those complex functions  $\phi(z)$ , analytic in  $\operatorname{Re}(z) > \alpha$ , bounded in each half plane  $\operatorname{Re}(z) \geq \alpha + \delta$ ,  $\delta > 0$ , and satisfying

$$\int_{-\infty}^{\infty} |\phi(\xi + i\eta)|^2 d\eta \leq M_{\phi}, \quad \xi > \alpha, \quad (2.23)$$

where  $M_\phi$  is a positive number depending only on  $\phi$  (and not, in particular, on  $\xi$ ). It is known (see, e.g. [10]) that each such function has a limiting "boundary" function

$$\phi_\alpha(\eta) = \lim_{\xi \rightarrow \alpha} \phi(\xi + i\eta) \quad (2.24)$$

defined almost everywhere in  $-\infty < \eta < \infty$  and  $\phi_\alpha(\eta)$  is measurable with

$$\int_{-\infty}^{\infty} |\phi_\alpha(\eta)|^2 d\eta \leq M_\phi.$$

Each  $\phi \in H_\alpha^2$  is the Laplace transform of a unique function  $f \in L_{loc}^2[0, \infty)$  such that

$$\int_0^\infty |e^{-\alpha t} f(t)|^2 dt < \infty.$$

Let  $\mu$  be a (non-negative valued) measure defined on the Borel subsets of  $\{z \mid z > \alpha\}$ . Then  $\mu$  is a Carleson measure if for every real  $\tau$  and every  $h > 0$

$$\mu(\{z \mid \tau - h \leq \operatorname{Im}(z) \leq \tau + h, \alpha < \operatorname{Re}(z) \leq \alpha + h\}) \leq Ah \quad (2.25)$$

for some positive  $A$  depending only on  $\mu$  (not on  $h$ ).

For a Carleson measure we have

**Theorem 2.4.** If  $\mu$  is a Carleson measure on  $\{z \mid \operatorname{Re}(z) > \alpha\}$  with  $A$  as in (2.25), if  $\phi \in H_\alpha^2$ , and  $\phi_\alpha$  is given by (2.24), then

$$\int_{\{z \mid \operatorname{Re}(z) > \alpha\}} |\phi(z)|^2 d\mu(z) \leq \frac{1000A}{\pi^2} \int_{-\infty}^{\infty} |\phi_\alpha(\eta)|^2 d\eta. \quad (2.26)$$

A proof of this theorem is offered, for the sake of completeness, in Section 4 of this paper. The relevance of this theorem for our present studies is exhibited in the selection of a particular measure  $\mu$ . For  $b \in Y'$  and a given discrete spectrum  $\{\lambda_k\}$  for  $A$ , let

$$\mu = \mu_{b, \{\lambda_k\}}$$

be defined by

$$\mu(-\lambda_k) = |b_k|^2, \quad k \in K, \quad (2.27)$$

$$\mu(\{z | \operatorname{Re}(z) > \alpha\} - \{\lambda_k | k \in K\}) = 0. \quad (2.28)$$

In this case the left hand side of (2.26) becomes

$$\sum_K |b_k \phi(-\lambda_k)|^2 \quad (\text{cf. (2.21)}).$$

The Plancherel Theorem, on the other hand, gives

$$\begin{aligned} \int_{-\infty}^{\infty} |\phi_{\alpha}(\eta)|^2 d\eta &= 2\pi \int_0^{\infty} |e^{-\alpha t} f(t)|^2 dt \\ &\leq 2e^{2|\alpha|T} \pi \int_0^T |f(t)|^2 dt \end{aligned}$$

when the support of  $f$  is restricted to  $[0, T]$ . Thus

$$\sum_K |b_k \phi(-\lambda_k)|^2 \leq 2000e^{2|\alpha|T} \frac{A}{\pi} \int_0^T |f(t)|^2 dt$$

and, in view of our earlier discussion, we have

Corollary 2.5. A sufficient condition in order that  $b \in Y'$  should be an  
admissible input element for the system (2.1), wherein  $\sigma(A) = \{\lambda_k | k \in K\}$   
and the corresponding eigenvectors  $\phi_k$ ,  $k \in K$ , form a strongly independent,  
uniformly  $l^2$ -convergent Schauder basis for  $X$ , is that the measure  
 $\mu_{b, \{\lambda_k\}}$  defined by (2.27), (2.28) should be a Carleson measure in

$\{z \mid \operatorname{Re}(z) > \alpha\}$  for some real  $\alpha$ .

We remark that the assumption (i) above together with the Hille-Yoshida Theorem ([4], [11]) implies that the complex numbers  $-\lambda_k$ ,  $k \in K$ , are, indeed, confined to some right half plane  $\operatorname{Re}(z) > \alpha$ . The fact that the support of  $f$  is restricted to  $[0, T]$  implies that the corresponding Laplace transform  $\phi$  is entire and satisfies an inequality (2.23) for every real  $\alpha$  ( $M_\phi = M_{\phi, \alpha}$  here).



### 3. Identification of Admissible and Inadmissible Input Elements; Examples .

Our first task in this section will be to develop a method whereby input elements  $b$  not in the state space  $X$  may be identified as particular elements of a larger space  $Y'$ . The assumptions made will be somewhat more restrictive than those introduced in Section 2. They are by no means necessary conditions.

Let us suppose that the operator  $A$ , generating a strongly continuous semigroup  $S(t)$  on the Hilbert space  $X$ , has (dense) domain  $\mathcal{D}(A)$  and that  $A$  possesses discrete eigenvalues  $\lambda_k$ ,  $k \in K$ , with

$$\lim_{\rho(k) \rightarrow \infty} |\lambda_k| = \infty.$$

Here  $\rho(k)$  denotes the number of elements  $l \in K$  such that  $l < k$  with respect to the assumed order relation on  $K$ . The corresponding normalized eigenvectors  $\phi_k$  are assumed to form a uniform basis for  $X$ . We denote the dual operator by  $A'$ . It has the same eigenvalues  $\lambda_k$  and the corresponding eigenvectors  $\psi_k$ ,  $k \in K$ , will be assumed normalized so that

$$\langle \psi_k, \phi_l \rangle = \begin{cases} 1, & k = l \\ 0, & k \neq l. \end{cases}$$

The  $\psi_k$  also form a uniform basis for  $X$ , as is well known. Then it is easy to see that

$$\mathcal{D}(A) = \{y = \sum_K x_k \phi_k \mid \sum_K |\lambda_k x_k|^2 < \infty\}$$

and that

$$\mathcal{D}(A') = \{y = \sum_K y_k \psi_k \mid \sum_K |\lambda_k y_k|^2 < \infty\}.$$

For the work of this section we take  $Y = \mathcal{D}(A')$  with the graph norm

$$\|y\|_Y^2 = \sum_K (1 + |\lambda_k|^2) |y_k|^2,$$

where

$$y = \sum_{k=1}^{\infty} y_k \psi_k$$

in  $X$ . Then  $Y \subset X$  and the injection mapping is continuous. It will often be possible to identify a Hilbert space  $Z \subset X$  with continuous injection map such that  $\|\cdot\|_Z$  is a familiar (e.g. Sobolev) norm and  $Y$  is a closed subspace of  $Z$  on which the norms  $\|\cdot\|_Z$  and  $\|\cdot\|_Y$  are equivalent.

We will be concerned with two different extensions of the operator  $A$ . We suppose first of all that there is an element  $\hat{x} \in X$  not in  $\mathcal{D}(A)$  and that  $L$  is an operator on  $X$  such that

$$\mathcal{D}(L) = \{\xi + u\hat{x} \mid \xi \in \mathcal{D}(A), u \text{ scalar}\},$$

$$Lx = Ax, \quad x \in \mathcal{D}(A).$$

We will refer to  $L$  as an "operational extension" of  $A$ . Its significance arises from the fact that many of the inhomogeneous boundary value problems arising in applications can be expressed in the form

$$\frac{dx}{dt} = Lx, \tag{3.1}$$

with the restriction

$$x = \xi + u\hat{x} \in \mathcal{D}(L). \tag{3.2}$$

The second extension of  $A$ , which is a map

$$\hat{A}: X \rightarrow Y',$$

is a standard one, often used, e.g. in [14]. If  $y, \eta \in \mathcal{D}(A)$ ,  $\mathcal{D}(A')$ , respectively, we have

$$\langle Ay, \eta \rangle = \langle y, A' \eta \rangle .$$

Since  $A' : Y = \mathcal{D}(A') \rightarrow X$  is continuous, the form  $\langle y, A' \eta \rangle$  extends to  $\langle x, A' \eta \rangle$ ,  $x \in X$ , by continuity and density of  $\mathcal{D}(A)$  in  $X$  and, so extended,  $\langle x, A' \eta \rangle$  defines, for each fixed  $x \in X$ , a continuous linear functional on  $Y$ , i.e., an element of  $Y'$ . We define

$$\hat{A} : X \rightarrow Y' = (\mathcal{D}(A'))'$$

by

$$\langle \hat{A}x, \eta \rangle = \langle x, A' \eta \rangle, \quad x \in X, \quad \eta \in Y = \mathcal{D}(A').$$

Our first goal, with reference to the system (3.1), (3.2), is to replace it by an infinite set of scalar ordinary differential equations

$$\frac{dx_k}{dt} = \lambda_k x_k + b_k u, \quad k \in K, \quad (3.3)$$

where

$$x(t) = \sum_{k \in K} x_k(t) \phi_k,$$

convergent in  $X$ . In order to do this we recognize first of all that

$$z = \sum_K \lambda_k x_k \phi_k$$

represents not  $Lx$ , but rather  $\hat{A}x$ , since

$$\langle \hat{A}x, \psi_k \rangle = \langle x, A' \psi_k \rangle = \langle x, \lambda_k \psi_k \rangle = \lambda_k x_k.$$

We rewrite (3.1) in the form

$$\frac{dx}{dt} = \hat{A}x + Lx - \hat{A}x, \quad (3.4)$$

an equation in  $Y'$ . Then, since  $x$  is to have the form (3.2) with  $\xi \in \mathcal{D}(A)$ , and since

$$L\xi = \hat{A}\xi = A\xi, \quad \xi \in \mathcal{D}(A),$$

(3.4) becomes

$$\frac{dx}{dt} = \hat{A}x + (\hat{L}x - \hat{A}x)u.$$

We define  $b \in Y'$ , a continuous linear functional on  $Y = \mathcal{D}(A')$ , by

$$\langle b, \eta \rangle = \langle \hat{L}x - \hat{A}x, \eta \rangle = \langle \hat{L}x, \eta \rangle - \langle \hat{x}, A'\eta \rangle \quad (3.5)$$

for  $\eta \in \mathcal{D}(A') \equiv Y$ . We then have

$$b = \sum_K b_k \phi_k$$

where the "control input coefficients",  $b_k$ , are given by

$$\begin{aligned} b_k &= \langle b, \psi_k \rangle = \langle \hat{L}x, \psi_k \rangle - \langle \hat{x}, A'\psi_k \rangle \\ &= \langle \hat{L}x, \psi_k \rangle - \lambda_k \langle \hat{x}, \psi_k \rangle. \end{aligned} \quad (3.6)$$

In most examples we shall have  $\hat{L}x = 0$ . Then, if

$$\hat{x} = \sum_K \hat{x}_k \phi_k,$$

convergent in  $X$ , we obtain, in place of (3.6),

$$b_k = -\lambda_k \hat{x}_k, \quad k \in K. \quad (3.7)$$

Also, in this case, the equation (3.5) becomes

$$\langle b, \eta \rangle = -\langle x, A'\eta \rangle. \quad (3.8)$$

The equation (3.5) (or (3.8)) will generally be used to identify the functional form of  $b$  while (3.6) (or (3.7)) will be used to identify its expansion coefficients in terms of the eigenvectors  $\phi_k$  of the operator  $A$ . While not all admissible input elements can be treated this way the class is large enough, we believe to warrant the detailed description we have given here.

Example 1: Heat Equation. Let  $x(s, t)$  satisfy

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}, \quad 0 < s < 1, \quad t > 0, \quad (3.9)$$

with boundary conditions

$$x(0, t) = 0, \quad \alpha x(1, t) + \beta \frac{\partial x}{\partial s}(1, t) = u(t), \quad (3.10)$$

where  $\alpha, \beta$  are real numbers, not both equal to zero. In this case we take

$$\begin{aligned} X &= L^2[0, 1] \\ Ax &= \frac{\partial^2 x}{\partial s^2}, \quad x \in \mathcal{D}(A) = \{x \in H^2[0, 1] \mid x(0) = 0, \\ &\quad \alpha x(1) + \beta x'(1) = 0\}, \\ Lx &= \frac{\partial^2 x}{\partial s^2}, \quad x \in \mathcal{D}(L) = \{x \in H^2[0, 1] \mid x(0) = 0\}, \\ \hat{x}(s) &= \begin{cases} \frac{s}{\alpha + \beta} & \alpha + \beta \neq 0 \\ \frac{s(2-s)}{\alpha} & \alpha + \beta = 0. \end{cases} \end{aligned} \quad (3.11)$$

With

$$\langle x, y \rangle = \int_0^1 x(s) y(s) ds$$

we see that if  $x, y \in \mathcal{D}(A)$

$$\begin{aligned} \langle Ax, y \rangle - \langle x, Ay \rangle &= \int_0^1 (x''(s) y(s) - x(s) y''(s)) ds \\ &= \int_0^1 \frac{d}{ds} (x'(s) y(s) - x(s) y'(s)) ds = (\text{since } x(0) = y(0) = 0) \\ &\quad x'(1) y(1) - x(1) y'(1) \\ &= \begin{cases} (x'(1) + \frac{\alpha}{\beta} x(1)) y(1) - x(1) (\frac{\alpha}{\beta} y(1) + y'(1)), & \beta \neq 0 \\ x'(1) (y(1) + \frac{\beta}{\alpha} y'(1)) - (x(1) + \frac{\beta}{\alpha} x'(1)) y'(1), & \alpha \neq 0 \end{cases} \\ &= 0 \end{aligned}$$

and we conclude  $A = A'$ . In the first case of (3.11),  $\alpha + \beta \neq 0$ ,  $\widehat{Lx} = 0$  and we have, for  $\eta \in \mathcal{D}(A') = \mathcal{D}(A)$

$$\begin{aligned} \langle b, \eta \rangle &= -\langle \widehat{x}, A'\eta \rangle = -\frac{1}{\alpha + \beta} \int_0^1 s \eta''(s) ds \\ &= \frac{1}{\alpha + \beta} \left( -s\eta'(s) \Big|_0^1 + \eta(s) \Big|_0^1 \right) = \frac{-\eta'(1) + \eta(1)}{\alpha + \beta} \\ &= \begin{cases} \frac{1}{\beta} \eta(1), & \beta \neq 0 \\ -\frac{1}{\alpha} \eta'(1), & \alpha \neq 0. \end{cases} \end{aligned}$$

Thus we have

$$b = \begin{cases} \frac{1}{\beta} \delta(1), & \beta \neq 0 \\ \frac{1}{\alpha} \delta'(1), & \alpha \neq 0. \end{cases} \quad (3.12)$$

The two agree if neither  $\alpha$  nor  $\beta$  are zero because the linear functional  $\frac{1}{\beta} \delta(1) + \frac{1}{\alpha} \delta'(1)$  is zero in  $(\mathcal{D}(A))' = Y'$  in this case.

The eigenvalues of  $A$  are  $\lambda_k = -\omega_k^2$  where, for  $k = 1, 2, 3, \dots$

$$\alpha \sin(\omega_k) + \beta \omega_k \cos(\omega_k) = 0. \quad (3.13)$$

Let

$$\begin{aligned} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2 \omega_k^2}} &= \sin \theta_k \\ \frac{\beta \omega_k}{\sqrt{\alpha^2 + \beta^2 \omega_k^2}} &= \cos \theta_k \end{aligned}$$

and (3.13) becomes

$$\cos(\omega_k - \theta_k) = 0$$

so that

$$\omega_k - \theta_k = \left( \frac{2k-1}{2} \right) \pi, \quad k = 1, 2, 3, \dots,$$

giving

$$\omega_k = \left(\frac{2k-1}{2}\right) \pi + \sin^{-1} \left( \frac{\alpha}{\sqrt{\alpha^2 + \beta^2 \omega_k^2}} \right).$$

It is easy to see that  $\frac{1}{\omega_k} = \mathcal{O}\left(\frac{1}{k}\right)$  as  $k \rightarrow \infty$  so

$$\omega_k = \left(\frac{2k-1}{2}\right) \pi + \mathcal{O}\left(\frac{1}{k}\right), \quad \beta \neq 0, \quad (3.14)$$

$$\omega_k = \left(\frac{2k-1}{2}\right) \pi + \frac{\pi}{2} = k\pi, \quad \beta = 0. \quad (3.15)$$

Defining

$$v_k^2 = \int_0^1 \sin(\omega_k s)^2 ds$$

it is easily seen that in all cases the  $v_k$  are nonzero and

$$\lim_{k \rightarrow \infty} v_k = \frac{1}{\sqrt{2}}.$$

Then the eigenfunctions

$$\phi_k(s) = \frac{1}{v_k} \sin(\omega_k s)$$

form an orthonormal basis for  $L^2[0, 1]$ . It follows then that the coefficients of the input distribution elements (3.12) are given by

$$b_k = \begin{cases} \frac{1}{\beta v_k} \sin(\omega_k), & \beta \neq 0, \\ \frac{\omega_k \cos(\omega_k)}{\alpha v_k}, & \alpha \neq 0. \end{cases} \quad (3.16)$$

We consider here the case  $\beta \neq 0$ , saving the analysis for  $\beta = 0$  until later in this section. If  $\beta \neq 0$ , formula (3.16) shows the  $b_k$  to be uniformly bounded. The complex numbers  $-\lambda_k = \omega_k^2$  have the property (from (3.14))

$$-\lambda_k = \left(\frac{2k-1}{2}\right)^2 \pi^2 + \mathcal{O}(1). \quad (3.17)$$

Thus the number of such  $-\lambda_k$  in any set  $|\operatorname{Im}(z) - \tau| \leq h$ ,  $\alpha \leq \operatorname{Re}(z) \leq \alpha + h$  is  $O(h^{1/2})$  and it follows that the measure  $\mu$  with  $\mu(-\lambda_k) = |b_k|^2$ ,  $\mu(\{\operatorname{Re}(z) \geq \alpha\} - \bigcup_{k=1}^{\infty} \{-\lambda_k\}) = 0$  is a Carleson measure. Hence if  $\beta \neq 0$  the boundary input (3.10) is admissible.

In this case the result is easily obtained without the Carleson measure theorem; for, if the coefficients  $c_k$  are square summable and  $T > 0$ ,

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} b_k c_k e^{\lambda_k t} \right\|_{L^2[0, T]} &\leq \sup_k \{|b_k|\} \sqrt{\sum_{k=1}^{\infty} |c_k|^2} \sqrt{\sum_{k=1}^{\infty} \int_0^T e^{2\lambda_k t} dt} \\ &\leq \sup_k \{|b_k|\} \sqrt{\sum_{k=1}^{\infty} |c_k|^2} \sqrt{\sum_{k=1}^{\infty} \int_0^{\infty} e^{2\lambda_k t} dt} = \sup_k \{|b_k|\} \sqrt{\sum_{k=1}^{\infty} |c_k|^2} \sqrt{\sum_{k=1}^{\infty} \frac{-1}{2\lambda_k}} < \infty \end{aligned} \quad (3.18)$$

since  $\sup_k \{|b_k|\} < \infty$ , and we conclude that the function sequence  $\{b_k e^{-\lambda_k t}\}$  is  $l^2$ -convergent in  $L^2[0, T]$ . Our next example is chosen in such a way that a simple argument of this type does not apply and the Carleson theorem is actually needed.

Example 2. Another Heat Conduction System. As a further example we ask the reader to consider the system shown in Figure 3.1.



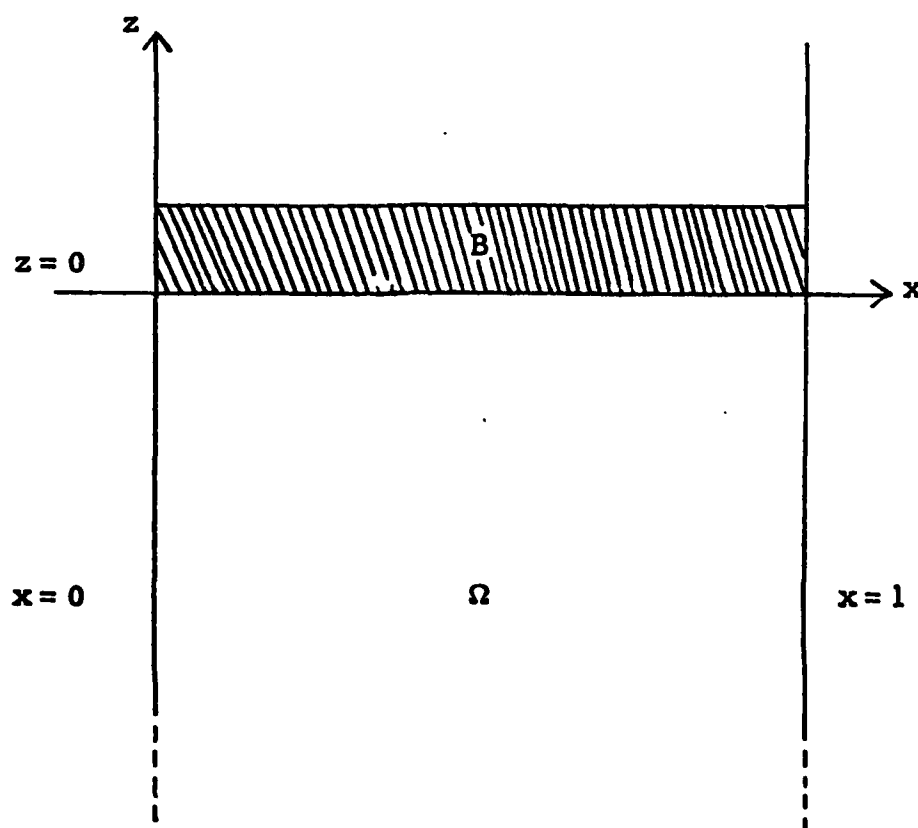


Figure 3.1

The shaded horizontal bar,  $B$ , represents a layer of material, whose depth will be assumed negligible, and whose heat conductivity,  $k$ , is small in comparison to its specific heat  $R$  while the region  $\Omega$  consisting of the half strip

$$\Omega : 0 \leq x \leq 1, z \leq 0,$$

is assumed filled with a material whose specific heat,  $r$ , is small by comparison with its conductivity,  $K$ . The heat flow equations are thus

$$R \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - K \frac{\partial \tau}{\partial z}, \quad (3.19)$$

$$r \frac{\partial \tau}{\partial t} = K \left( \frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial z^2} \right), \quad (3.20)$$

together with boundary conditions

$$\frac{\partial T}{\partial x}(0, t) = 0, \quad \frac{\partial T}{\partial x}(1, t) = 0 \quad (3.21)$$

$$\frac{\partial \tau}{\partial x}(0, z, t) = 0, \quad \frac{\partial \tau}{\partial x}(1, z, t) = g(z) u(t), \quad (3.22)$$

$$\lim_{z \rightarrow -\infty} \frac{\partial \tau}{\partial z}(x, z, t) = \lim_{z \rightarrow -\infty} \tau(x, z, t) = 0 \quad (3.23)$$

$$\tau(x, 0, t) = T(x, t), \quad 0 \leq x \leq 1. \quad (3.24)$$

The inhomogeneous boundary condition along  $x = 1, z \leq 0$ , represents the input heat flux. In (3.19), (3.21),  $T(x, t)$  is the temperature in the bar,  $\tau(x, z, t)$  the temperature in  $\Omega$ .

If we assume  $k, r$  very small by comparison with  $R, K$ , we may, as an idealization, replace (3.19) and (3.20) by

$$R \frac{\partial T}{\partial t} = -K \frac{\partial \tau}{\partial z} \quad (3.25)$$

$$\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial z^2} = 0, \quad (3.26)$$

retaining the boundary conditions (3.21) - (3.24). We take as our basic state space

$$\mathcal{T} = \{T = T(x) \mid T \in L^2[0, 1]\}.$$

We define an operator  $A$  on  $\mathcal{T}$  with domain

$$\mathcal{D}(A) = H^1[0, 1]$$

as follows . Given  $T \in \mathcal{D}(A)$ , we let  $\tau = \tau(x, z)$  satisfy (3.26) in  $\Omega$  together with

$$\tau(x, 0) = T(x), \quad 0 \leq x \leq 1, \quad (3.27)$$

and

$$\frac{\partial \tau}{\partial x}(0, z) = 0, \quad \frac{\partial \tau}{\partial x}(1, z) = 0 \quad (3.28)$$

$$\lim_{z \rightarrow -\infty} \frac{\partial \tau}{\partial z}(\cdot, z) = 0 \quad \text{in } L^2[0, 1]. \quad (3.29)$$

$$\lim_{z \rightarrow -\infty} \tau(\cdot, z) = 0 \quad \text{in } H^1[0, 1]. \quad (3.30)$$

From [14], for  $T \in H^1[0, 1]$  we have  $\tau \in H^{3/2}(\Omega)$ . The trace theorem ([1], [14]) then gives

$$\frac{\partial \tau}{\partial z}(\cdot, 0) \in L^2[0, 1],$$

and we define

$$AT = -\frac{K}{R} \frac{\partial \tau}{\partial z}(\cdot, 0). \quad (3.31)$$

So doing, (3.25) becomes

$$\dot{T} = AT \quad (3.32)$$

and (3.31) is subsumed in the definition of  $A$ .

Lemma 3.1. The operator  $-A$  is the positive square root of the Sturm-Liouville operator

$$ST = -\frac{K^2}{R^2} \frac{d^2 T}{dx^2}$$

with

$$\mathfrak{S}(T) = \{T \in H^2[0, 1] \mid \frac{dT}{dx}(0) = \frac{dT}{dx}(1) = 0\}.$$

Proof. We compute  $(-A)^2 T$  for  $T \in \mathfrak{S}(T)$ . For such  $T$  the solution of (3.27) - (3.30)  $\in H^{5/2}(\Omega)$ . If we let

$$\hat{\tau}(x, z) = \frac{K}{R} \frac{\partial \tau}{\partial z}(x, z)$$

then

$$\hat{\tau}(\cdot, 0) = -AT$$

and

$$\begin{aligned} (-A)^2 T &= -A \hat{\tau}(\cdot, 0) = \frac{K^2}{R^2} \frac{\partial^2 \tau}{\partial z^2}(\cdot, 0) \\ &= -\frac{K^2}{R^2} \frac{\partial^2 \tau}{\partial x^2}(\cdot, 0) = -\frac{K^2}{R^2} \frac{d^2 T}{dx^2} \end{aligned}$$

since  $\tau \in H^{5/2}(\Omega)$  together with (3.26) implies that

$$\frac{\partial^2 \tau}{\partial x^2}(\cdot, 0) + \frac{\partial^2 \tau}{\partial z^2}(\cdot, 0) = 0 \quad \text{in } L^2[0, 1]$$

and  $T = \tau(\cdot, 0)$ .

The positivity of  $-A$  follows from the divergence theorem. If  $T \in \mathfrak{S}(A)$  and  $\tau = \tau(x, z)$  is constructed as above, we have

$$\begin{aligned} &\iint_{\Omega} \left[ \left( \frac{\partial \tau}{\partial x}(x, z) \right)^2 + \left( \frac{\partial \tau}{\partial z}(x, z) \right)^2 \right] dx dz \\ &= \iint_{\Omega} \|\nabla \tau(x, z)\|^2 dx dz \quad (\nabla = \text{gradient}) \\ &= \iint_{\Omega} [\operatorname{div}(\tau(x, z) \nabla \tau(x, z)) - \tau(x, z) \Delta^2 \tau(x, y)] dx dz \\ &(\Delta^2 = \text{Laplacian}) = (\text{from (3.26)}) \\ &= \iint_{\Omega} \operatorname{div}(\tau(x, z) \nabla \tau(x, z)) dx dz = (\text{using (3.27) - (3.30)}) \\ &= \int_0^1 \tau(x, 0) \frac{\partial \tau}{\partial z}(x, 0) dx = (T, -AT)_{L^2[0, 1]}. \end{aligned}$$

This completes the proof .

Accordingly,  $A$  is self adjoint with eigenfunctions

$$\phi_0(x) \equiv 1, \quad \phi_k(x) = \sqrt{2} \cos(k\pi x), \quad k = 1, 2, 3, \dots \quad (3.33)$$

and eigenvalues

$$\lambda_0 = 0, \quad \lambda_k = -\frac{K}{R} k\pi, \quad k = 1, 2, 3, \dots \quad (3.34)$$

Let  $w(x, z)$  be the solution of the following inhomogeneous boundary value problem:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} &= 0 \quad \text{in } \Omega \\ \frac{\partial w}{\partial x}(0, z) &= 0, \quad \frac{\partial w}{\partial x}(1, z) = g(z) \\ \lim_{z \rightarrow -\infty} \frac{\partial w}{\partial z}(x, z) &= \lim_{z \rightarrow -\infty} w(x, z) = 0. \\ w(x, 0) &\equiv 0, \quad 0 \leq x \leq 1. \end{aligned}$$

We will assume that  $g(z)$  is such that the resulting  $w(x, z) \in H^2(\Omega)$ .

In this case the inhomogeneous equation can be interpreted as

$$\dot{T} = AT + bu$$

where  $b = b(x)$  is given by

$$b(x) = -\frac{K}{R} \frac{\partial w}{\partial z}(x, 0).$$

To compute the coefficients of the expansion

$$b(x) = \sum_{k=0}^{\infty} b_k \phi_k(x),$$

we note that since  $A$  is self adjoint,  $\psi_k(x) = \phi_k(x)$ , and

$$b_k = \int_0^1 \phi_k(x) b(x) dx. \quad (3.35)$$

Let  $\phi_k(x, z)$  be the solution of

$$\frac{\partial^2 \phi_k}{\partial x^2} + \frac{\partial^2 \phi_k}{\partial z^2} = 0 \quad \text{in } \Omega$$

with

$$\phi_k(x, 0) = \phi_k(x)$$

and homogeneous boundary conditions of the type (3.27) - (3.30) otherwise.

Then, with  $\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ ,

$$\begin{aligned} 0 &= \int_{\Omega} [\phi_k(x, z) \Delta^2 w(x, z) - w(x, z) \Delta^2 \phi_k(x, z)] dx dz \\ &= \int_{\Omega} \operatorname{div}[\phi_k(x, z) \operatorname{grad} w(x, z) - w(x, z) \operatorname{grad} \phi_k(x, z)] dx dz \end{aligned}$$

$$= -\frac{R}{K} \int_0^1 \phi_k(x) b(x) dx + \int_{-\infty}^0 \phi_k(1, z) g(z) dz$$

giving (cf. (3.35))

$$b_k = \frac{K}{R} \int_{-\infty}^0 \phi_k(1, z) g(z) dz.$$

Now it is easily checked that for  $k = 1, 2, 3, \dots$

$$\phi_k(x, z) = (\sqrt{2} \cos k\pi x) (\exp(k\pi z))$$

so that

$$\phi_k(1, z) = (-1)^k \sqrt{2} \exp(k\pi z)$$

and thus

$$b_k = \frac{(-1)^k \sqrt{2} K}{R} \int_{-\infty}^0 \exp(k\pi z) g(z) dz$$

The Carleson measure theorem can be used in a slightly different way than that set forth in Corollary 2.5 to show that if  $g \in L^2(-\infty, 0]$  then the  $b_k$  are square summable and  $b$  is, consequently an element of  $L^2[0, 1]$ . Writing  $\zeta = -z$ ,  $g(-\zeta) = \tilde{g}(\zeta)$ , we see that

$$b_k = \frac{(-1)^k \sqrt{2} K}{R} \int_0^\infty \exp(-k\pi \zeta) \tilde{g}(\zeta) d\zeta.$$

Since the measure  $\mu$  assigning the value 1 to each of the points  $k\pi$ ,  $k = 0, 1, 2, \dots$ , is clearly a Carleson measure, and since  $\frac{(-1)^k \sqrt{2} K}{R}$  changes only in sign,  $\{b_k\} \in l^2$ .

If  $g(z)$  is just bounded and measurable on  $-\infty < z \leq 0$  we can almost trivially obtain

$$b_k = O\left(\frac{1}{k}\right)$$

and the  $b_k$  will be square summable.

It is obviously possible to replace  $g(z)$  by distributions of various types. Taking  $g(z) = \delta_{(0)}$  corresponds to a point heat source at the corner  $x = 1$ ,  $z = 0$  and leads to

$$b_k = \frac{(-1)^k \sqrt{2} K}{R} \quad (3.36)$$

In our present example  $X = L^2[0, 1]$ ,  $Y = \mathfrak{g}(A) = H^1[0, 1]$  and  $Y' = H^{-1}[0, 1]$ . The coefficients (3.36) may be recognized as those corresponding to  $\delta_{(1)}$  (referring now to distributions along the  $x$ -axis).

Any measure  $\mu$  assigning to the points  $-\lambda_k = \frac{K}{R} k\pi$  values  $|b_k|^2$  which are bounded evidently yields a Carleson measure and we conclude that all of the above cases correspond to admissible input elements. In this case the argument represented by the inequalities (3.18) will not work because the series  $\sum_{k=1}^{\infty} \left(\frac{-1}{2\lambda_k}\right)$  is not summable in this example.

Example 3. Hyperbolic and Neutral Systems.

A wide variety of systems involving linear hyperbolic partial differential equations in two independent variables  $x, t$ , or neutral functional equations lead to systems of the form described at the beginning of this section, the eigenvectors,  $\phi_k$ , of  $A$  forming a uniform  $l^2$ -basis for the state space  $X$  and the eigenvalues  $\lambda_k$  confined to a vertical strip  $\alpha < \operatorname{Re}(\lambda) < \beta$  in the complex plane. It also usually turns out in these cases that the number of  $\lambda_k$  in any rectangle

$$\alpha < \operatorname{Re}(\lambda) < \beta, \quad \gamma < \operatorname{Im}(\lambda) < \delta$$

is less than or equal to  $M(\delta - \gamma)$ , where  $M$  is a fixed positive number. It is evident that the measure (2.27), (2.28) is a Carleson measure in these cases whenever the control input coefficients  $b_k$  constitute a bounded set.

Example 4. Linear Surface Waves. If the operator  $A$  is defined as in (3.31) but, instead of the first order system (3.32) we consider the second order counterpart

$$\ddot{\xi} + A \dot{\xi} = 0 \tag{3.37}$$

we obtain the linearized equations for small amplitude waves on the surface of an incompressible fluid. The theory is more fully developed in [16], [17], [19]. With  $\eta = \dot{\xi}$ , (3.37) is equivalent to the first order system

$$\begin{pmatrix} \dot{\xi} \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv G \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{3.38}$$

To obtain a topology corresponding to the energy of the system one defines



$$\left\| \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \right\|_e^2 = \|\zeta\|_{L_0^2[0,1]}^2 + (\eta, A^{-1}\eta)_{L_0^2[0,1]} \quad (3.39)$$

where

$$L_0^2[0,1] = \{ \zeta \in L^2[0,1] \mid \int_0^1 \zeta(x) dx = 0 \}. \quad (3.40)$$

The restriction to  $L_0^2[0,1]$  corresponds to conservation of fluid volume.

On the domain

$$\mathfrak{D}_0(A) = \{ \zeta \in H^1[0,1] \mid \int_0^1 \zeta(x) dx = 0 \}$$

the operator  $A$  is invertible. Its eigenvalues are (cf. (3.34))

$$\lambda_k = -\frac{K}{R} k\pi, \quad k = 1, 2, 3, \dots \quad (3.41)$$

with the same eigenfunctions  $\phi_k(x)$ ,  $k = 1, 2, 3, \dots$ , as shown in (3.33). Correspondingly, the operator  $G$  has eigenvalues

$$i\omega_k, -i\omega_k, \quad \omega_k = \frac{K\pi}{R} k^{1/2} \equiv \gamma k^{1/2}, \quad k = 1, 2, 3, \dots \quad (3.42)$$

and the eigenvectors, orthonormalized with respect to  $\|\cdot\|_e$  and the corresponding inner product are

$$\psi_k = \begin{pmatrix} \phi_k \\ i\omega_k \phi_k \end{pmatrix}, \quad \psi_{-k} = \begin{pmatrix} \phi_k \\ -i\omega_k \phi_k \end{pmatrix}, \quad k = 1, 2, 3, \dots \quad (3.43)$$

To discuss admissible input elements in this case we let  $\beta_k, \beta_{-k}$  be non-negative numbers,  $k = 1, 2, 3, \dots$ , and define

$$\begin{aligned} \mu\{i\omega_k\} &= \beta_k, \quad \mu\{-i\omega_k\} = \beta_{-k}, \quad k = 1, 2, 3, \dots, \\ \mu(\{ \operatorname{Re}(z) \geq \alpha \} - \bigcup_{k=1}^{\infty} (\{i\omega_k\} \cup \{-i\omega_k\})) &= 0. \end{aligned}$$

Let  $\beta(\omega)$ ,  $-\infty < \omega < \infty$ , be defined as the piecewise linear function such that in the interval  $[\omega_k, \omega_{k+1}]$

$$\beta(\omega) = \frac{\beta_k(\omega_{k+1} - \omega) + \beta_{k+1}(\omega - \omega_k)}{\omega_{k+1} - \omega_k} \quad (3.44)$$

Since

$$\begin{aligned} \int_{\omega_k}^{\omega_{k+1}} \omega^{1/2} \beta(\omega) d\omega &= \frac{\beta_k \omega_k^{1/2} + \beta_{k+1} \omega_{k+1}^{1/2}}{2} [\omega_{k+1} - \omega_k] \\ &\rightarrow \frac{1}{4} (\beta_k + \beta_{k+1}), \quad k \rightarrow \infty, \end{aligned}$$

we conclude that  $\mu$  is a Carleson measure just in case there is a constant  $C$  such that

$$\int_{\sigma}^{\tau} \omega^{1/2} \beta(\omega) d\omega \leq C |\tau - \sigma| \quad (3.45)$$

whenever  $0 < \sigma < \tau$ , together with a comparable condition involving the  $\beta_{-k}$  and negative values of  $\omega$ . But (3.45) is true just in case

$$\omega_k^{1/2} \beta_k \leq C, \quad k = 1, 2, 3, \dots$$

and the comparable condition for negative  $k$  is

$$\omega_k^{1/2} \beta_{-k} \leq C, \quad k = 1, 2, 3, \dots$$

Thus for the inhomogeneous system

$$\begin{pmatrix} \dot{\zeta} \\ \eta \end{pmatrix} = a \begin{pmatrix} \zeta \\ \eta \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u$$

the input element  $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$  with

$$\left\langle \begin{pmatrix} \phi_k \\ 1\omega_k \phi_k \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\rangle = \beta_k$$

$$\left\langle \begin{pmatrix} \phi_k \\ -1\omega_k \phi_k \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\rangle = \beta_{-k}$$

is admissible, from this criterion, if

$$k^{1/2} (|\beta_k|^2 + |\beta_{-k}|^2) < C \quad (3.46)$$

for some fixed positive number  $C$ . It will be noted that this is (slightly) less restrictive than the requirement

$$\left\| \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|_e < \infty.$$

**Example 5. Negative Results.** For any system similar to the one in Example 4 but with  $|\omega_{k+1} - \omega_k| = O(1/k^{1/2} + \varepsilon)$  the Carleson measure condition will be stronger than requiring  $b \in X$ . Hence failure of the Carleson measure condition cannot be used to show that an element  $b$  is not admissible, for any  $b \in X$  is admissible.

To illustrate what can be done in a negative direction, we return to Example 1 with  $\beta = 0$ . This situation has been studied, using a different approach, in [13]. We present here an argument more in the spirit of the present work. As shown in (3.15),

$$\lambda_k = -\omega_k^2 = -k^2 \pi^2 \quad (3.47)$$

and (cf. (3.16) and, w.l.o.g. taking  $\alpha = 1$ )

$$b_k = \sqrt{2} k\pi \cos(k\pi) = (-1)^k \sqrt{2} k\pi. \quad (3.48)$$

Since  $\beta_k = |b_k|^2 = 2k^2 \pi^2$  while  $(k+1)^2 \pi^2 - k^2 \pi^2 = 2k\pi^2 + \pi^2$ , it is not hard to see that the measure  $\mu = \mu_{b, \{\lambda_k\}}$ ,  $\mu(-\lambda_k) = |b_k|^2$  is not a Carleson measure in this case. As we have remarked, this by itself is not enough to show that the input element with coefficients (3.48) is not admissible. To show this, we ask the reader to consider the function

$$\psi_r(z) = (z+1)^{-r}$$

analytic in the complex plane minus the cut consisting of  $\{z \mid z \text{ real}, z \leq -1\}$ .

If  $r > \frac{1}{2}$ ,  $\psi_r$  is square integrable on any vertical line  $\{z \mid \operatorname{Re}(z) = \xi, \xi \geq 0\}$  with uniformly bounded  $L^2$  norm and  $\psi_r(z)$  is bounded for  $\operatorname{Re}(z) \geq 0$ . It follows that  $\psi_r(z)$  is the Laplace transform of a function  $f_r = f_r(t)$  with  $f_r \in L^2[0, \infty)$ . Then

$$\begin{aligned} b_k \int_0^\infty e^{-k^2 \pi^2 t} f_r(t) dt \\ = (-1)^k \sqrt{2} k\pi \psi_r(k^2 \pi^2) &= \frac{(-1)^k \sqrt{2} k\pi}{(k^2 \pi^2 + 1)^r} \\ = o(|k|^{1-2r}), \quad k \rightarrow \infty. \end{aligned} \quad (3.49)$$

This expression is not square summable if  $r$  satisfies the inequalities

$$1 - 2r \geq -\frac{1}{2},$$

so we require

$$\frac{1}{2} < r \leq \frac{3}{4}.$$

Let  $E$  be the closed subspace spanned by the functions  $e^{-k^2 \pi^2 t}$  in  $L^2[0, \infty)$  and let  $E_T$ ,  $T > 0$ , be the subspace of  $L^2[0, T]$  consisting of restrictions to  $[0, T]$  of functions in  $E$ . If  $\hat{f}_r$  is the orthogonal projection of  $f_r$  onto  $E$  we clearly have

$$\int_0^\infty e^{-k^2 \pi^2 t} \hat{f}_r(t) dt = \int_0^\infty e^{-k^2 \pi^2 t} f_r(t) dt.$$

It is shown in [7], [18] that the natural restriction map  $R: E \rightarrow E_T$  is onto, (obviously) bounded and (not so obviously) boundedly invertible with respect to the induced  $L^2[0, \infty)$ ,  $L^2[0, T]$  topologies of  $E$ ,  $E_T$ , respectively. Thus, with  $p_k(t) = e^{-k^2 \pi^2 t}$ ,

$$\begin{aligned} \int_0^\infty e^{-k^2 \pi^2 t} \hat{f}_r(t) dt &= (\hat{f}_r, p_k)_{L^2[0, \infty)} \\ &= (\hat{f}_r, R^{-1} R p_k)_{L^2[0, \infty)} = ((R^{-1})^* \hat{f}_r, R p_k)_{L^2[0, T]} \\ &= \int_0^T e^{-k^2 \pi^2 t} \varphi_r(t) dt \end{aligned}$$

where

$$\varphi_r = (R^{-1})^* \hat{f}_r \in E_T \subset L^2[0, T].$$

It follows that  $\varphi_r$  is an element of  $L^2[0, T]$  such that the numbers

$$(-1)^k \sqrt{2} k \pi \int_0^T e^{-k^2 \pi^2 t} \varphi_r(t) dt, \quad k = 1, 2, 3, \dots$$

are not square summable. From earlier developments, the input element  $b$  with coefficients (3.12) corresponding to the boundary condition (3.10), with  $\beta = 0$ ,  $\alpha = 1$ :

$$x(1, t) = u(t),$$

is not an admissible input element.

4. A Proof of Theorem 2.4. It is clear that the Carleson measure theorem in  $H^2_\alpha$ , Theorem 2.4, is central to our work in this paper. This result, in one form or another has been known for somewhat more than a decade. A proof for  $H^2(D)$ , where  $D$  is the unit disc in the complex plane, appears in Duren [5]. A proof for functions in  $H^1_\alpha$  is given, by Koosis in his recent book [12]. The reader is also referred to the recent book [8] by J. Garnett. Because the result is not particularly well known outside the circle of mathematicians working in  $H^p$  theory and because the results are rather scattered and not readily available in precisely the form we require, we offer here a proof of Theorem 2.4 which is a direct adaptation to the half plane of the result for the unit disc appearing in Duren's book. The proof given here originally formed part of the first author's doctoral dissertation [9]. As in Duren's work, the proof makes use of a relatively simple case of the Marcinkiewicz interpolation theorem ([23], Chapter XI) and, again following Duren, we do not quote the general Marcinkiewicz theorem but, rather, give a direct proof for the simple special case required here.

We begin with a covering lemma of "Vitali type".

Lemma 4.1. Let  $\{I_\lambda \mid \lambda \in \Lambda\} \equiv \mathcal{J}$  be a family of intervals in  $\mathbb{R}^1$ . Suppose there is a positive number  $K$  such that for any finite collection  $\{I_{\lambda_1}, I_{\lambda_2}, \dots, I_{\lambda_n}\}$  of disjoint intervals in  $\mathcal{J}$

$$\sum_{k=1}^n |I_{\lambda_k}| < K. \quad (4.1)$$

Then we can choose a sequence  $\{I_{\lambda_k} \mid k = 1, 2, 3, \dots\}$  of disjoint intervals  
from  $J$  with the property: for every  $\lambda \in \Lambda$  there exists  
 $k \in \{1, 2, 3, \dots\}$  such that

$$I_{\lambda} \subset I_k$$

where  $I_k$  is the interval having the same center as  $I_{\lambda_k}$  but five times  
the length of  $I_{\lambda_k}$ .

Proof. From (4.1) it follows, in particular, that the length,  $|I_{\lambda}|$ , of  
 $I_{\lambda}$  is uniformly bounded (take  $n = 1$ ,  $\lambda_1 = \lambda$ ). Define the sequence  
 $\{I_{\lambda_k}\}$  inductively as follows. Let  $I_{\lambda_1}$  be such that

$$|I_{\lambda_1}| \geq \frac{1}{2} \sup_{\lambda \in \Lambda} |I_{\lambda}|.$$

For  $k = 2, 3, 4, \dots$  let  $I_{\lambda_k}$  be disjoint from  $I_{\lambda_i}$ ,  $i = 1, 2, \dots, k-1$ ,  
and such that

$$|I_{\lambda_k}| \geq \frac{1}{2} \sup\{|I_{\lambda}| \mid \lambda \in \Lambda, I_{\lambda} \cap I_{\lambda_i} = \emptyset, i = 1, 2, \dots, k-1\}. \quad (4.2)$$

Since the  $I_{\lambda_k}$  are disjoint it follows from (4.1) that

$$\lim_{k \rightarrow \infty} |I_{\lambda_k}| = 0. \quad (4.3)$$

Let  $I_{\lambda} \in J$ . Then there exists  $k$  such that

$$I_{\lambda} \cap I_{\lambda_k} \neq \emptyset. \quad (4.4)$$

Otherwise (4.2) and (4.3) could not both be true. Let  $k_0$  be the  
smallest integer such that (4.4) is true. Then

$$|I_\lambda| \leq 2 |I_{\lambda_{k_0}}|$$

and, together with the fact that  $I_\lambda \cap I_{\lambda_{k_0}} \neq \emptyset$ , this implies that  $I_\lambda \subset I_{k_0}$ , completing the proof.

We subdivide the rest of the proof of Theorem 2.4 into several propositions for clarity. The proof is given for the half plane  $\operatorname{Re}(z) > 0$ , without loss of generality, and we designate  $H_0^2$  simply by  $H^2$ .

Proposition 4.2. Let  $\phi \in H^2$  and let  $\phi_0(i \cdot)$  be the corresponding boundary function in  $L^2(-\infty, \infty)$ . For  $z = \sigma + i\tau$ ,  $\sigma > 0$ , let  $I_z$  be the interval

$$I_z = [\tau - \sigma, \tau + \sigma] \quad (4.5)$$

and let

$$\tilde{\phi}(z) = \sup_{I \in \mathcal{J}_z} \frac{1}{|I|} \int_I |\phi_0(it)| dt, \quad (4.6)$$

where  $\mathcal{J}_z$  is the set of all finite intervals containing  $I_z$ . Then

$$|\phi(z)| \leq \frac{10}{\pi} \tilde{\phi}(z). \quad (4.7)$$

Proof. From the Poisson integral formula in the half plane we have

$$\phi(z) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma \phi_0(it) dt}{\sigma^2 + (\tau - t)^2}$$

so that

$$|\phi(z)| \leq \frac{1}{\pi} \left[ \sum_{N=0}^{\infty} \int_{2^N \sigma \leq |t-\tau| \leq 2^{N+1} \sigma} \frac{\sigma |\phi_0(it)| dt}{\sigma^2 + (\tau - t)^2} + \int_{|t-\tau| < \sigma} \frac{\sigma |\phi_0(it)| dt}{\sigma^2 + (\tau - t)^2} \right]$$



$$\begin{aligned}
&\leq \frac{1}{\pi} \left[ \sum_{N=0}^{\infty} \int_{|t-\tau| \leq 2^{N+1}\sigma} \frac{|\phi_0(it)| dt}{4^N \sigma} + \int_{|t-\tau| \leq \sigma} \frac{|\phi_0(it)| dt}{\sigma} \right] \\
&\leq \frac{1}{\pi} \left[ \sum_{N=0}^{\infty} \frac{1}{2^{N-2}} \tilde{\phi}(z) + 2 \tilde{\phi}(z) \right] \\
&= \frac{10}{\pi} \tilde{\phi}(z).
\end{aligned}$$

Proposition 4.3.      Let  $\psi \in L^1(-\infty, \infty)$  and, for  $z = \sigma + i\tau$ ,  
 $\sigma > 0$ , let  $I_z$  be given by (4.5) while (cf. (4.6))

$$\tilde{\psi}(z) = \sup_{I \in \mathcal{J}_z} \frac{1}{|I|} \int_I |\psi(t)| dt. \quad (4.8)$$

Let  $\mu$  be a Carleson measure and; for  $s \geq 0$  let  $E_s$  be the Borel  
measurable subset of  $\{z \mid \operatorname{Re}(z) > 0\}$  given by

$$E_s = \{z \mid \operatorname{Re}(z) > 0, \tilde{\psi}(z) > s\}.$$

Then, with  $A$  as in Definition 2.3,

$$\mu(E_s) \leq \frac{5A}{2s} \|\psi\|_{L^1(-\infty, \infty)}. \quad (4.9)$$

Proof.      Let  $\mathcal{J}$  be the family of all finite intervals in  $\mathbb{R}^1$  such that

$$\frac{1}{|I|} \int_I |\psi(t)| dt > s. \quad (4.10)$$

If  $I_1, I_2, \dots, I_n \in \mathcal{J}$  are disjoint, then (4.10) gives, for every  $n$ ,

$$\sum_{k=1}^n |I_k| \leq \frac{1}{s} \sum_{k=1}^n \int_{I_k} |\psi(t)| dt \leq \frac{1}{s} \|\psi\|_{L^1(-\infty, \infty)}. \quad (4.11)$$

Thus  $\mathcal{J}$  satisfies the hypotheses of Lemma 4.1 and we can find a disjoint  
sequence  $\{I_n \mid n = 1, 2, 3, \dots\} \subset \mathcal{J}$  such that,  $I_n$  having the same

center as  $I_n$  but five times the length, each  $I \in \mathcal{J}$  is contained in some  $J_n$ .

If  $z \in E_S$ , then  $I_z \subset I$  for some  $I \in \mathcal{J}$  and we have, for some  $n$ ,

$$I_z = [\tau - \sigma, \tau + \sigma] \subset J_n.$$

Then clearly,

$$z \in S_n = \{\sigma + i\tau \mid 0 < \sigma \leq \frac{|J_n|}{2}, \tau \in J_n\}.$$

This being true for all  $z \in E_S$ ,

$$E_S \subset \bigcup_{n=1}^{\infty} S_n.$$

Since  $\mu$  is a Carleson measure and (4.11) holds,

$$\begin{aligned} \mu(E_S) &\leq \sum_{n=1}^{\infty} \mu(S_n) \leq A \sum_{n=1}^{\infty} \frac{|J_n|}{2} \\ &= \frac{5A}{2} \sum_{n=1}^{\infty} |I_n| \leq \frac{5A}{2s} \|\psi\|_{L^1(-\infty, \infty)}. \end{aligned}$$

**Proposition 4.4.** Let  $\phi \in H_0^2$  with boundary function  $\phi_0(\cdot) \in L^2(-\infty, \infty)$ . Let  $\phi(z)$  be defined by (4.6). Then, if  $\mu$  is a Carleson measure on  $\{z \mid \operatorname{Re}(z) > 0\}$ ,

$$\int_{\operatorname{Re}(z) > 0} (\tilde{\phi}(z))^2 d\mu(z) \leq 10A \int_{-\infty}^{\infty} |\phi_0(it)|^2 dt. \quad (4.12)$$

**Proof.** For each  $r > 0$  let

$$\psi_r(t) = \begin{cases} \phi_0(it) & \text{if } |\phi_0(it)| > r \\ 0 & \text{otherwise} \end{cases}.$$

From  $\phi_0(1 \cdot) \in L^2(-\infty, \infty)$ , we conclude that the support of  $\psi_r$  is a subset  $\Sigma_r$  of  $(-\infty, \infty)$  of finite (Lebesgue) measure. Then  $\psi_r \in L^2(\Sigma_r) \subset L^1(\Sigma_r)$  and we conclude, since  $\psi_r$  vanishes outside  $\Sigma_r$ , that  $\psi_r \in L^1(-\infty, \infty)$ . Moreover

$$\begin{aligned} \int_0^\infty \|\psi_r\|_{L^1(-\infty, \infty)} dr &= \int_0^\infty \int_{\Sigma_r} |\phi_0(it)| dt dr \\ &= \int_{-\infty}^\infty \int_0^\infty dr |\phi_0(it)| dt = \int_{-\infty}^\infty |\phi_0(it)|^2 dt \\ &= \|\phi_0(1 \cdot)\|_{L^2(-\infty, \infty)}^2. \end{aligned} \quad (4.13)$$

Let  $\alpha(s) = \mu(E_s)$ . Then we can see that

$$\int_{\operatorname{Re}(z) > 0} (\tilde{\rho}(z))^2 d\mu(z) = - \int_0^\infty s^2 d\alpha(s) = 2 \int_0^\infty s \alpha(s) ds. \quad (4.14)$$

From the definition (4.8) of  $\tilde{\psi}$  it is clear that for any two such functions,  $\psi_1, \psi_2$ , we have

$$(\psi_1 + \psi_2)(z) \leq \tilde{\psi}_1(z) + \tilde{\psi}_2(z).$$

Hence

$$\begin{aligned} \tilde{\rho}(z) &= (\psi_r + (\tilde{\rho}_0(1 \cdot) - \psi_r))(z) \\ &\leq \tilde{\psi}_r(z) + \overline{(\phi_0(1 \cdot) - \psi_r)}(z) \\ &\leq \tilde{\psi}_r(z) + r \end{aligned} \quad (4.15)$$

since  $|\phi(it) - \psi_r(t)|$  is either equal to 0 or is  $\leq r$ . Let

$$F_s = \{z \mid \tilde{\psi}_r(z) > s\}.$$

Suppose  $z \in E_{2r}$ . Then  $\tilde{\phi}(z) > 2r$  and (4.15) gives

$$\tilde{\psi}_r(z) \geq \tilde{\phi}(z) - r > r$$

and we conclude  $z \in F_r$ . Thus

$$E_{2r} \subset F_r.$$

Hence, from (4.9) of Proposition 4.3,

$$\mu(E_r) \leq \mu(F_{r/2}) \leq \frac{5A}{r} \|\psi_r\|_{L^1(-\infty, \infty)}$$

so that

$$\begin{aligned} \int_0^\infty r \alpha(r) dr &= \int_0^\infty r \mu(E_r) dr \\ &\leq 5A \int_0^\infty \|\psi_r\|_{L^1(-\infty, \infty)} dr \leq (\text{using (4.13)}) \\ &\leq 5A \|\phi_0(i \cdot)\|_{L^2(-\infty, \infty)}^2. \end{aligned}$$

Then (4.14) gives the inequality (4.12).

The proof of Theorem 2.4 is completed by combining (4.7) of Proposition 4.2 with (4.12) above to give

$$\begin{aligned} \int_{\operatorname{Re}(z) > 0} |\phi(z)|^2 d\mu(z) &\leq \frac{100}{\pi^2} \int_{\operatorname{Re}(z) > 0} (\tilde{\phi}(z))^2 d\mu(z) \\ &\leq \frac{1000A}{\pi^2} \int_{-\infty}^\infty |\phi_0(it)|^2 dt \end{aligned}$$

as claimed in (2.26), except for the trivial detail of replacing  $\phi_0(1 \cdot)$  by  $\phi_\alpha = \phi(\alpha + 1 \cdot)$ .

### References

- [1] Adams, R. A.: "Sobolev Spaces", Academic Press, New York, 1975.
- [2] Carleson, Lennart: "On infinite differential equations with constant coefficients, I", Math. Scand. 1 (1953), pp. 31-38.
- [3] Curtain, R., and A. Pritchard: "Infinite Dimensional Linear Systems Theory", Lecture Notes on Control and Information Sciences, Vol. 8, Springer-Verlag, New York 1978.
- [4] Dunford, N., and J. T. Schwartz: "Linear Operators, Part I: General Theory", Interscience Pub. Co., New York, 1957.
- [5] Duren, P. L.: "Theory of  $H^p$  Spaces", Academic Press, New York 1970.
- [6] Fattorini, H. O.: "Boundary control systems", SIAM J. on Control 6 (1968), pp. 349-385.
- [7] Fattorini, H. O., and D. L. Russell: "Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations", Quart. Appl. Math., 1974, pp. 45-69.
- [8] Garnett, J.: "Bounded Analytic Functions", Academic Press, New York 1981.
- [9] Ho, L. F.: "Controllability and spectral assignability of a class of hyperbolic control systems with retarded control canonical forms", Thesis, University of Wisconsin, Madison, August 1981.
- [10] Hoffmann, K.: "Banach Spaces of Analytic Functions", Prentice-Hall, Englewood Cliffs, N. J., 1962.

- [11] Kato, T. : "Perturbation Theory for Linear Operators", Springer-Verlag, New York, 1966 .
- [12] Koosis, P. : "Introduction to  $H$  Spaces", Cambridge Univ. Press, New York, 1980 .
- [13] Lions, J. L. : "Optimal Control of Systems Governed by Partial Differential Equations", tr. by S. K. Mitter, Springer-Verlag, New York, 1971.
- [14] Lions, J. L., and E. Magenes : "Non-homogeneous boundary value problems", Vols. I, II, Springer-Verlag, New York, 1971.
- [15] Rabinowitz, P. H. : "Periodic solutions of nonlinear hyperbolic partial differential equations", Comm. Pur. Appl. Math., 20 (1967), pp. 145-204.
- [16] Reid, R. M., Thesis, University of Wisconsin, Madison, August 1979.
- [17] Reid, R. M. and D. L. Russell : "Water waves and problems of infinite time control", Proc. IRIA . Symp. on Analysis and Control of Systems, Rocquencourt, Dec. 1978.
- [18] Schwartz, L. : "Etude des sommes d'exponentielles", deuxieme edition, Hermann, Paris, 1959 .
- [19] Stoker, J. J. : "Water Waves", Interscience Pub. Co., New York, 1957.
- [20] Washburn, D. C. : "A semigroup theoretic approach to modelling of boundary input problems", Proc. IFIP Conf. on Modelling and Identification of Distributed Parameter Systems, Rome, 1976 .

- [ 21 ] Young, R. M. : "An Introduction to Nonharmonic Fourier Series",  
Academic Press, San Francisco, 1980 .
- [ 22 ] Zabczyk, J. : "A semigroup approach to boundary value control",  
Proc. 2nd IFAC Symposium on Distributed Parameter Systems, Warwick ,  
June 1977 .
- [ 23 ] Zygmund, A. : "Trigonometric Series", Cambridge Univ. Press,  
Cambridge, 1968 .



**END**

**FILMED**

**1-84**

**DTIC**